

# Diffeomorphisms groups of tame Cantor sets and Thompson-like groups

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## Abstract

The group of  $\mathcal{C}^1$ -diffeomorphisms of any sparse Cantor subset of a manifold is countable and discrete (possibly trivial). Thompson's groups come out of this construction when we consider central ternary Cantor subsets of an interval. Brin's higher dimensional generalizations  $nV$  of Thompson's group  $V$  arise when we consider products of central ternary Cantor sets. We derive that the  $\mathcal{C}^2$ -smooth mapping class group of a sparse Cantor sphere pair is a discrete countable group and produce this way versions of the braided Thompson groups.

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## 1 Introduction

Differentiable structures on Cantor sets have first been considered by Sullivan in [34]. Our aim is to consider groups of diffeomorphisms of Cantor sets, mapping class groups of Cantor punctured spheres and their relations with Thompson-like groups. In particular, the usual Thompson groups (see [10]) can be retrieved as diffeomorphisms groups of Cantor subsets of suitable spaces (a line, a circle or a 2-sphere).

Let  $M$  be a compact manifold and  $C \subset M$  be a Cantor set, namely a *compact totally disconnected subset without isolated points*. Any two Cantor sets are homeomorphic as topological spaces. But if  $M$  has dimension  $m \geq 3$  there exist Cantor sets  $C_1$  and  $C_2$  embedded into  $M$  so that there is no ambient homeomorphism of  $M$  carrying  $C_1$  into  $C_2$ . One says that  $C_1$  and  $C_2$  are not *topologically equivalent* Cantor set embeddings.

A Cantor subset of  $\mathbb{R}^m$  is *tame* if there is a homeomorphism of  $\mathbb{R}^m$  which sends it within a coordinates axis. All Cantor sets in  $\mathbb{R}^m$ , for  $m \leq 2$  are tame, but there exist uncountably many *wild* (i.e. not tame) Cantor sets in  $\mathbb{R}^m$ , for every  $m \geq 3$  (see [3]).

One defines similarly *smooth equivalence* and *smoothly tame* Cantor sets. The analogous story for diffeomorphisms is already interesting for  $m = 1$ , as Cantor subsets of  $\mathbb{R}$  might be differentiably non-equivalent. Our main concern is the image of the group of diffeomorphisms of  $M$  which preserve a Cantor set  $C$  into the automorphism group of  $C$ . Under fairly general conditions we are able to prove that this is a countable group, thereby providing an interesting class of discrete groups. For Cantor sets obtained from a topological iterated function system the associated groups are non-trivial, while for many self-similar Cantor sets these are versions of Thompson's groups.

## A. General countability statements

### 1.1 Pure mapping class groups

**Definition 1.** Let  $M$  be a compact orientable manifold and  $C \subset M$  a Cantor subset. We denote by  $\text{Diff}^k(M, C)$  the group of diffeomorphisms of class  $\mathcal{C}^k$  of  $M$  sending  $C$  to itself, by  $\text{Diff}^{k,+}(M, C)$  the subgroup of orientation preserving diffeomorphisms and by  $\text{PDiff}^{k,+}(M, C)$  the subgroup of pure orientation preserving diffeomorphisms, i.e. pointwise preserving  $C$ .

The  $\mathcal{C}^k$ -mapping class group  $\mathcal{M}^{k,+}(M, C)$  is the group  $\pi_0(\text{Diff}^{k,+}(M, C))$  of  $\mathcal{C}^k$ -isotopy classes of orientation preserving diffeomorphisms rel  $C$  (i.e. which are identity on  $C$ ) of class  $\mathcal{C}^k$ . The pure  $\mathcal{C}^k$ -mapping class group  $PM^{k,+}(M, C)$  is the group  $\pi_0(\text{PDiff}^{k,+}(M, C))$  of  $\mathcal{C}^k$ -isotopy classes of pure orientation preserving  $\mathcal{C}^k$ -diffeomorphisms rel  $C$ .

In a similar vein but a different context, the group of homeomorphisms  $\text{Diff}^0(M, A)$  associated to a manifold  $M$  and a countable dense set  $A \subset M$  was studied recently in [14]. The authors proved there that  $\text{Diff}^0(M, A)$  is either isomorphic to a countably infinite product of copies of  $\mathbb{Q}$ , when  $M$  is 1-dimensional, or the Erdős subgroup of  $l^2$  elements, otherwise. In the present setting, when  $A$  is closed and the smoothness is at least  $\mathcal{C}^1$  the situation is fundamentally different.

If we write  $C = \bigcap_{j=1}^{\infty} C_j$ , where each  $C_j$  is a compact submanifold of  $M$  and  $C_{j+1} \subset \text{int}(C_j)$  for all  $j$ , then the sequence  $\{C_j\}$  is called a *defining sequence* for  $C$ . It is known that  $C$  is a tame Cantor set if and only if we can choose  $C_j$  to be finite unions of disjoint disks.

**Definition 2.** The class of  $\varphi$  in  $PM^{k,+}(M, C)$  is compactly supported if there exists some defining sequence  $\{C_j\}$  of  $C$  and some  $n$  for which the restriction of  $\varphi$  to  $C_n$  is  $\mathcal{C}^k$ -isotopic to identity rel  $C$ .

Note that the property of being compactly supported is independent of the choice of the defining sequence. Our first result is the following:

**Theorem 1.** Let  $C$  be a  $\mathcal{C}^k$ -tame Cantor set, namely a Cantor subset of a closed interval  $\mathcal{C}^k$ -embedded in a compact orientable manifold  $M$  of dimension at least 2. If  $k \geq 2$ , then all classes in the group  $PM^{k,+}(M, C)$  are compactly supported. In particular, the group  $PM^{k,+}(M, C)$  is countable.

In contrast, the topological mapping class group  $PM^{0,+}(S^2, C)$  is uncountable. We might expect  $PM^{k,+}(M, C)$  be countable for  $k \geq 2$  even when  $C$  is only a  $\mathcal{C}^0$ -tame Cantor subset of  $M$ .

The following precises the second statement in Theorem 1:

**Corollary 1.** Let  $C$  be a  $\mathcal{C}^k$ -tame Cantor subset of a compact orientable surface  $M$  and  $\{C_j\}$  be a defining sequence for  $C$  consisting of finite unions of disjoint disks. If  $k \geq 2$ , then  $PM^{k,+}(M, C)$  coincides with the inductive limit  $\lim_{j \rightarrow \infty} PM^{k,+}(M - \text{int}(C_j))$  of pure mapping class groups of compact subsurfaces.

Note that, when  $N$  is a compact surface the isomorphism type of  $PM^{k,+}(N)$  is independent of  $k$ .

### 1.2 $\mathcal{C}^1$ -diffeomorphisms groups of Cantor sets

We now turn to the full mapping class groups. Several groups which arose recently in the literature could be thought to play the role of the mapping class groups for some infinite type surfaces, for instance the group  $\mathcal{B}$  from [17] and its version  $BV$ , which was defined by Brin [6] and Dehornoy [12], independently. These two groups are braidings of the Thompson group  $V$  (see [10]). Geometric constructions of the same sort permitted the authors of [18] to derive two braidings  $T^*$  and  $T^\#$  of the Thompson group  $T$ .

Our next goal is to show that these groups are indeed smooth mapping class groups in the usual sense and that most (if not all) smooth mapping class groups of Cantor sets are related to some Thompson-like groups.

Let assume for the moment that  $C \subset M$  is smoothly tame. Set then  $\mathfrak{diff}_M^k(C)$  and  $\mathfrak{diff}_M^{k,+}(C)$  for the groups of induced transformations of  $C$  arising as restrictions of elements of  $\text{Diff}^k(M, C)$  and  $\text{Diff}^{k,+}(M, C)$ , respectively. The  $\mathcal{C}^k$  topology on  $\text{Diff}^k(M, C)$  induces a topology on  $\mathfrak{diff}_M^k(C)$ .

Notice now that we have the exact sequence:

$$1 \rightarrow P\mathcal{M}^{k,+}(M, C) \rightarrow \mathcal{M}^{k,+}(M, C) \rightarrow \mathfrak{diff}_M^{k,+}(C) \rightarrow 1. \quad (1)$$

By Theorem 1 the group  $\mathcal{M}^{k,+}(M, C)$  is discrete countable if and only if  $\mathfrak{diff}_M^{k,+}(C)$  does, when  $k \geq 2$  and  $M$  is compact (or the interior of a compact manifold).

Classical Thompson groups can be realized as groups of dyadic piecewise linear homeomorphisms (or bijections) of an interval, circle or a Cantor set (see [10, 19]) or as groups of automorphisms at infinity of graphs (respecting or not the planarity), as in [28]. Notice that the more involved construction from [19] provides embeddings of Thompson groups into the group of diffeomorphisms of the circle, admitting invariant minimal Cantor sets. In particular, Ghys and Sergiescu obtained embeddings as discrete subgroups of the group of diffeomorphisms (see [19], Thm. 2.3).

In our setting we see that whenever it is discrete and countable the group  $\mathcal{M}^{k,+}(M, C)$  is the braiding of  $\mathfrak{diff}_M^{k,+}(C)$  according to Corollary 1, as in the cases studied in [6, 12, 17, 18]. This pops out the question whether  $\mathfrak{diff}_M^{k,+}(C)$  is a Thompson-like group, in general. We were not able to solve this question in full generality and actually when  $C$  is a generic Cantor set of the interval we expect the group  $\mathfrak{diff}_M^{k,+}(C)$  be trivial. To this purpose we introduce the following property of Cantor sets.

**Definition 3.** *The Cantor subset  $C$  of an interval is  $\sigma$ -sparse if, for any  $a, b \in C$ , there is a complementary interval  $(\alpha, \beta) \subset (a, b) \cap \mathbb{R} \setminus C$  such that*

$$\beta - \alpha \geq \sigma(b - a). \quad (2)$$

*Moreover  $C$  is sparse if it is  $\sigma$ -sparse for some  $\sigma > 0$ .*

Set  $\mathfrak{diff}^k(C) = \mathfrak{diff}_{\mathbb{R}}^k(C)$ ,  $\mathfrak{diff}^{k,+}(C) = \mathfrak{diff}_{\mathbb{R}}^{k,+}(C)$  for the sake of notational simplicity.

**Theorem 2.** *If  $C$  is a sparse Cantor subset of  $\mathbb{R}$ , then the group  $\mathfrak{diff}^1(C)$  is countable. If  $C$  is a sparse Cantor set in  $S^1 = \mathbb{R}/\mathbb{Z}$ , then  $\mathfrak{diff}_{S^1}^1(C)$  is countable.*

Theorem 2 cannot be extended to all Cantor sets  $C$ , without additional assumptions, as we can see from the examples given in section 5.

We have the following more general version of the previous result:

**Theorem 3.** *If  $C$  is a sparse Cantor subset of an interval  $\mathcal{C}^1$ -embedded into a compact orientable manifold  $M$ , then the group  $\mathfrak{diff}_M^1(C)$  is countable and discrete.*

Although we only considered smoothly tame Cantor subsets above, there is a large supply of topologically tame Cantor subsets in any dimensions for which we can prove the countability:

**Theorem 4.** *Let  $C_i$  be sparse Cantor sets in  $\mathbb{R}$  and  $C = C_1 \times C_2 \times \cdots \times C_n \subset \mathbb{R}^n$ . Then the group  $\mathfrak{diff}_{\mathbb{R}^n}^1(C)$  is countable.*

The key point is to show that the stabilizer of a point in this group is a finitely generated abelian group (see Lemma 7, Proposition 1). The discreteness of the stabilizers seems to be the counterpart to the following unpublished theorem of G. Hector (see [27]): If the subgroup  $G$  of the group  $\text{Diff}^\omega(S^1)$  of analytic diffeomorphisms of the circle has an exceptional minimal set, then the stabilizer  $G_a$  of any point  $a$  of the circle in  $G$  is either trivial or  $\mathbb{Z}$ . As a corollary every subgroup of  $\text{Diff}^\omega(S^1)$  having a minimal Cantor set is countable. This is, of course, not true for subgroups of  $\text{Diff}^\infty(S^1)$ . The proof of our result is related to Thurston's generalization of Reeb's stability theorem from [35].

## B. Specific families of Cantor sets

### 1.3 Iterated functions systems

**Definition 4.** *A contractive iterated function system (abbreviated contractive IFS) is a finite family  $\Phi = (\phi_0, \phi_1, \dots, \phi_n)$  of contractive maps  $\phi_j : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . Recall that a map  $\phi$  is contractive if its Lipschitz constant is smaller to unit, namely:*

$$\sup_{x, y \in \mathbb{R}^d} \frac{d(\phi(x), \phi(y))}{d(x, y)} < 1.$$

According to Hutchinson (see [23]) there exists a unique non-empty compact  $C = C_\Phi \subset \mathbb{R}^d$ , called the *attractor* of the IFS  $\Phi$ , such that  $C = \bigcup_{j=0}^n \phi_j(C)$ .

**Example 1.** The central Cantor set  $C_\lambda$ , with  $\lambda > 2$ , is the attractor of the IFS  $\Phi = (\phi_0, \phi_1)$  on  $\mathbb{R}$  given by

$$\phi_0(x) = \frac{1}{\lambda}x, \quad \phi_1(x) = \frac{1}{\lambda}x + \frac{\lambda-1}{\lambda}.$$

Although the IFS makes sense also when  $1 < \lambda \leq 2$ , in this case the attractor is not a Cantor set but the whole interval  $[0, 1]$ .

Consider now the following type of IFS of topological nature.

**Definition 5.** Let  $U$  be an orientable manifold (possibly non-compact) and  $\phi_j : U \rightarrow U$  be finitely many orientation preserving homeomorphisms on their image. We say that  $\Phi = (\phi_0, \phi_1, \dots, \phi_n)$  has a strict attractive basin  $M$  if  $M$  is a compact orientable submanifold  $M \subset U$  with the following properties:

1.  $\phi_j(M) \subset \text{int}(M)$ , for all  $j \in \{0, 1, \dots, n\}$ ;
2.  $\phi_i(M) \cap \phi_j(M) = \emptyset$ , for any  $j \neq i \in \{0, 1, \dots, n\}$ .

We say that the pair  $(\Phi, M)$  is an invertible IFS if  $M$  is a strict attractive basin for  $\Phi$ . If moreover,  $\phi_j$  are  $C^k$ -diffeomorphisms on their image, then we say that the IFS is of class  $C^k$ .

The existence of an attractive basin is a topological version of uniform contractivity of  $\phi_j$ . There exists then a unique invariant non-empty compact  $C_\Phi \subset M$  with the property that  $C_\Phi = \bigcup_{i=0}^n \phi_i(C_\Phi)$ .

**Theorem 5.** Consider a  $C^1$  contractive invertible IFS  $(\Phi, M)$ ,  $\Phi = (\phi_0, \phi_1, \dots, \phi_n)$ , whose strict attractive basin  $M$  is diffeomorphic to a  $d$ -dimensional ball. Then, the group  $\text{diff}_M^1(C_\Phi)$  contains the Thompson group  $F_{n+1}$ , when  $M$  is of dimension  $d = 1$  and the Thompson group  $V_{n+1}$ , when  $d \geq 2$ , respectively.

In particular, the groups  $\text{diff}_M^1(C_\Phi)$  are (highly) nontrivial.

For a clear introduction to the classical Thompson groups  $F, T, V$  we refer to [10]. The generalized versions  $F_n, T_n, V_n$  were considered by Higman ([22]) and further extended and studied by Brown and Stein (see [33]), Bieri and Strebel (see [2]) and Laget [25]. We will recall their definitions in section 2.

The result of the theorem does not hold when the attractive basin  $M$  is not a ball. For instance, when  $M$  is a 3-dimensional solid torus, by taking nontrivial (linked) embeddings  $\bigcup_{i=0}^n \phi_i(M) \subset M$  we can provide examples of wild Cantor sets, some of them being topologically rigid, in which case the group  $\text{diff}_M^1(C)$  is trivial (see [32, 36]).

## 1.4 Self-similar Cantor subsets of the line

The second part of this paper is devoted to concrete examples of groups arising by these constructions, for particular choices of Cantor sets. We will be concerned in this section with self-similar Cantor sets, namely attractors of IFS which consist only of similitudes. The typical example is the central ternary Cantor set  $C_\lambda \subset [0, 1]$  of parameter  $\lambda > 2$  from Example 1.

Let  $\Phi = (\phi_0, \phi_1, \dots, \phi_n)$  be an IFS of affine transformations of  $[0, 1]$ , given by:

$$\phi_j(x) = \lambda_j x + a_j,$$

where

$$0 = a_0 < \lambda_0 < a_1 < \lambda_1 + a_1 < a_2 < \dots < \lambda_{n-1} + a_{n-1} < a_n < \lambda_n + a_n = 1.$$

The last condition means that the segments  $\phi_j([0, 1])$  are mutually disjoint, so that the attractor  $C = C_\Phi$  is a sparse Cantor subset of  $[0, 1]$ . The positive reals  $g_j = a_{j+1} - \lambda_j - a_j$  are the initial *gaps* as they represent the distance between consecutive intervals  $\phi_j([0, 1])$  and  $\phi_{j+1}([0, 1])$ . The image of  $[0, 1]$  by the elements of the monoid generated by  $\Phi$  are called *standard intervals*.

We consider the groups  $F_C$  and  $T_C$  defined as follows. Let  $PL(\mathbb{R}, C)$  and  $PL(S^1, C)$  be the groups of orientation preserving piecewise linear homeomorphisms of  $\mathbb{R}$  and  $S^1 = \mathbb{R}/\mathbb{Z}$  respectively, keeping invariant

$C$ , i.e. of those  $\varphi$  for which there exists a finite covering of  $C$  by standard disjoint intervals  $\{I_j\}$ ,  $\mathbf{k}_j \in \mathbb{Z}^{n+1}$  and  $a_j, b_j \in C$ , such that

$$\varphi(x) = b_j + \Lambda_{\mathbf{k}_j}(x - a_j), \quad \text{for any } x \in I_j, \quad (3)$$

where  $\Lambda_{\mathbf{k}} = \prod_{i=0}^n \lambda_i^{k_i}$ , for each multi-index  $\mathbf{k} = (k_0, k_1, \dots, k_n) \in \mathbb{Z}^{n+1}$ . Eventually  $F_C$  and  $T_C$  are the images of  $PL(\mathbb{R}, C)$  and  $PL(S^1, C)$ , respectively, in the group of homeomorphisms of  $C$ . Similarly we have the group of piecewise affine exchanges  $PE(C)$  which are (not necessarily orientation preserving) left continuous bijections of  $S^1$  preserving  $C$  as above. We denote by  $V_C$  its image into the group of homeomorphisms of  $C$ .

**Definition 6.** *The self-similar Cantor set  $C \subset [0, 1]$  satisfies the genericity condition (C) if*

1. *either all homothety ratios  $\lambda_i$  are equal and all initial generation gaps  $g_\alpha$  are equal;*
2. *or the factors  $\lambda_i$  and the gaps  $g_\alpha$  are incommensurable, in the following sense:*
  - (a)  $\Lambda_{\mathbf{k}} g_\alpha = g_\beta$  *implies that  $\mathbf{k} = 0$  and  $\alpha = \beta$ ;*
  - (b) *there exists no permutation  $\sigma$  different from identity and  $\mathbf{k}, \mathbf{k}_\alpha \in \mathbb{Z}_+^{n+1}$  such that for all  $\alpha$  we have:*

$$\frac{g_{\sigma(\alpha)}}{g_\alpha} = \Lambda_{-\mathbf{k} + \frac{1}{n} \sum_{\alpha=1}^n \mathbf{k}_\alpha}.$$

**Theorem 6.** *Let  $C \subset [0, 1]$  be a self-similar Cantor set satisfying the genericity condition (C). Then for every  $\varphi \in \text{diff}^{1,+}(C)$  we can find a covering of  $C$  by a finite collection of disjoint standard intervals  $\{I_j\}$ , whose images are also standard intervals, integers  $\mathbf{k}_j \in \mathbb{Z}^{n+1}$  and  $a_j, b_j \in C$ , such that the restriction of the map  $\varphi$  has the form*

$$\varphi(x) = b_j + \Lambda_{\mathbf{k}_j}(x - a_j), \quad \text{for any } x \in I_j \cap C. \quad (4)$$

*In particular,  $\text{diff}^{1,+}(C)$  is isomorphic to  $F_C$ ,  $\text{diff}_{S^1}^{1,+}(C)$  is isomorphic to  $T_C$  and  $\text{diff}_{S^2}^{1,+}(C)$  is isomorphic to  $V_C$ . Moreover, these are isomorphic to the Thompson groups  $F_{n+1}, T_{n+1}$  and  $V_{n+1}^\pm$ , respectively.*

The main points in the statement of the theorem are the finiteness of the covering and the fact that the intervals are standard. If we drop the requirement that the intervals be standard then a similar result holds with the same proof without the genericity condition (C) in the hypothesis.

We derive easily now the following interpretation for the Thompson groups and their braided versions:

**Corollary 2.** 1. *Let  $C$  be the image of the standard ternary Cantor subset into the equatorial circle of the sphere  $S^2$  and  $k \geq 2$ .*

- (a) *The smooth mapping class group  $\mathcal{M}^{k,+}(D_+^2, C)$  is the Thompson group  $T$ , where  $D_+^2$  is the upper hemisphere;*
- (b) *The smooth mapping class group  $\mathcal{M}^{k,+}(S^2, C)$  is the braided Thompson group  $\mathcal{B}$  from [17] (see section 2 for definitions).*

2. *Let  $C$  be the standard ternary Cantor subset of an interval contained in the interior of a 2-disk  $D^2$  and  $k \geq 2$ . Then  $\mathcal{M}^{k,+}(D^2, C)$  coincides with the braided Thompson group  $BV$  of Brin and Dehornoy.*

**Remark 1.** *The central ternary Cantor sets  $C_\lambda$  are pairwise non-diffeomorphic, i.e. there is no  $\mathcal{C}^1$  diffeomorphism of  $\mathbb{R}$  sending  $C_\lambda$  into  $C_{\lambda'}$  for  $\lambda \neq \lambda'$ . Indeed, if it were such a diffeomorphism then the Hausdorff dimensions of the two Cantor sets would agree, while the Hausdorff dimension of  $C_\lambda$  is  $\frac{\log 2}{\log \lambda}$  (see [15], Thm. 1.14). Nevertheless, the groups  $\text{diff}^{1,+}(C_\lambda)$  are all isomorphic, for  $\lambda > 2$ , according to Theorem 6.*

We notice that a weaker version of our Theorem 6 concerning the form of  $\mathcal{C}^1$ -diffeomorphisms of the central Cantor sets  $C_\lambda$ , was already obtained in ([1], Proposition 1).

A case which attracted considerable interest is that of bi-Lipschitz homeomorphisms of Cantor sets (see [11, 16] and the recent [29, 37]). In particular, the results of Falconer and Marsh [16] imply that every bi-Lipschitz homeomorphism of a Cantor set is given by a pair of possibly infinite coverings of the Cantor set by disjoint intervals and affine homeomorphisms between the corresponding intervals. Notice that any countable subgroup of  $\text{Diff}^0(S^1)$  (or  $\text{Diff}^0([0, 1])$ ) can be conjugated (by a homeomorphism) into the corresponding group of bi-Lipschitz homeomorphisms (see [13], Thm. D).

## 1.5 Self-similar Cantor dusts

The next step is to go to higher dimensions. Examples of Blankenship (see [3]) show that there exist wild Cantor sets in  $\mathbb{R}^n$ , for every  $n \geq 3$ . A Cantor set  $C$  is tame if and only if for every  $\varepsilon > 0$  there exist finitely many disjoint piecewise linear cells of diameter smaller than  $\varepsilon$  whose interiors cover  $C$ . In particular, products of tame Cantor sets are tame. More generally, the product of a Cantor subset of  $\mathbb{R}^n$  with any compact 0 dimensional subset  $Z \subset \mathbb{R}^m$  is a tame Cantor subset of  $\mathbb{R}^{m+n}$  (see [26], Cor.2).

In order to emphasize the role of the embedding we will consider now the simplest Cantor subsets, which although tame they are not smoothly tame. Let  $C_\lambda^n \subset \mathbb{R}^n$  be the Cartesian product of  $n$  copies of  $C_\lambda$ , where  $n \geq 2$  and  $\lambda > 2$ , which is itself a Cantor set.

**Theorem 7.** *Let  $\varphi \in \text{diff}_{\mathbb{R}^n}^{1,+}(C_\lambda^n)$ , where  $\lambda > 2$ . Then there is a covering of  $C_\lambda^n$  by a finite collection of disjoint standard parallelepipeds  $\{I_j\}$ , integers  $k_{j,i} \in \mathbb{Z}$  and  $a_{j,i}, b_{j,i} \in C_\lambda$ , such that:*

$$\varphi(x) = (b_{j,i} + \lambda^{k_{j,i}}(x_i - a_{j,i}))_{i=1,n} \circ S_j, \quad \text{for any } x \in I_j \cap C_\lambda^n. \quad (5)$$

where  $S_j$  is an orientation preserving symmetry of the cube. In particular,  $\text{diff}_{\mathbb{R}^n}^{1,+}(C_\lambda^n)$  is isomorphic to the symmetry extension  $nV^{\text{sym}}$  of Brin's higher dimensional Thompson group  $nV$  (see section 2).

Notice that in a series of papers (see [5, 7, 4, 21]) by Brin, Bleak and Lanoue, Hennig and Matucci the authors proved that  $nV$  are pairwise non-isomorphic finitely presented simple groups (see also [30, 31]).

**Remark 2.** *Note that the group  $\text{diff}_{[0,1]^n}^1(C_\lambda^n)$  is a proper subgroup of  $\text{diff}_{\mathbb{R}^n}^1(C_\lambda^n)$ .*

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## 2 Definition of Thompson-like groups

The standard reference for the classical Thompson groups is [10]. For the sake of completeness we provide here the basic definitions from several different perspectives, which lead naturally the path to the generalizations considered by Brown and Stein and further to the high dimensional Brin groups.

### 2.1 Groups of piecewise affine homeomorphisms/bijections

*Thompson's group  $F$*  is the group of piecewise dyadic affine homeomorphisms of the interval  $[0, 1]$ . Namely, for each  $f \in F$ , there exist two dyadic subdivisions of  $[0, 1]$ ,  $a_0 = 0 < a_1 < \dots < a_n = 1$  and  $b_0 = 0 < b_1 < \dots < b_n$ , with  $n \in \mathbb{N}^*$ , i.e. such that  $a_{i+1} - a_i$  and  $b_{i+1} - b_i$  belong to  $\{\frac{n}{2^k}, n, k \in \mathbb{N}\}$ , so that the restriction of  $f$  to  $[a_i, a_{i+1}]$  is the unique increasing affine map onto  $[b_i, b_{i+1}]$ .

Therefore, an element of  $F$  is completely determined by the data of two dyadic subdivisions of  $[0, 1]$  having the same cardinality.

Let us identify the circle to the quotient space  $[0, 1]/0 \sim 1$ . *Thompson's group  $T$*  is the group of piecewise dyadic affine orientation preserving homeomorphisms of the circle. In other words, for each  $g \in T$ , there exist two dyadic subdivisions of  $[0, 1]$ ,  $a_0 = 0 < a_1 < \dots < a_n = 1$  and  $b_0 = 0 < b_1 < \dots < b_n$ , with  $n \in \mathbb{N}^*$ , and  $i_0 \in \{1, \dots, n\}$ , such that, for each  $i \in \{0, \dots, n-1\}$ , the restriction of  $g$  to  $[a_i, a_{i+1}]$  is the unique increasing map onto  $[b_{i+i_0}, b_{i+i_0+1}]$ . The indices must be understood modulo  $n$ .

Therefore, an element of  $T$  is completely determined by the data of two dyadic subdivisions of  $[0, 1]$  having the same cardinality, say  $n \in \mathbb{N}^*$ , plus an integer  $i_0 \bmod n$ .

Finally, *Thompson's group  $V$*  is the group of bijections of  $[0, 1]$ , which are right-continuous at each point, piecewise nondecreasing and dyadic affine. In other words, for each  $h \in V$ , there exist two dyadic subdivisions of  $[0, 1]$ ,  $a_0 = 0 < a_1 < \dots < a_n = 1$  and  $b_0 = 0 < b_1 < \dots < b_n$ , with  $n \in \mathbb{N}^*$ , and a permutation  $\sigma \in \mathfrak{S}_n$ ,



such that, for each  $i \in \{1, \dots, n\}$ , the restriction of  $h$  to  $[a_{i-1}, a_i[$  is the unique nondecreasing affine map onto  $[b_{\sigma(i)-1}, b_{\sigma(i)}[$ . It follows that an element  $h$  of  $V$  is completely determined by the data of two dyadic subdivisions of  $[0, 1]$  having the same cardinality, say  $n \in \mathbb{N}^*$ , plus a permutation  $\sigma \in \mathfrak{S}_n$ . Denoting  $I_i = [a_{i-1}, a_i]$  and  $J_i = [b_{i-1}, b_i]$ , these data can be summarized into a triple  $((J_i)_{1 \leq i \leq n}, (I_i)_{1 \leq i \leq n}, \sigma \in \mathfrak{S}_n)$ .

We have obvious inclusions  $F \subset T \subset V$ . R.J. Thompson proved in 1965 that  $F, T$  and  $V$  are finitely presented groups and that  $T$  and  $V$  are simple (cf. [10]). The group  $F$  is not perfect, as  $F/[F, F]$  is isomorphic to  $\mathbb{Z}^2$ , but  $F' = [F, F]$  is simple. However,  $F'$  is not finitely generated (this is related to the fact that an element  $f$  of  $F$  lies in  $F'$  if and only if its support is included in  $]0, 1[$ ).

## 2.2 Groups of diagrams of finite binary trees

A *finite binary rooted planar tree* is a finite planar tree having a unique 2-valent vertex, called the *root*, a set of monovalent vertices called the *leaves*, and whose other vertices are 3-valent. The planarity of the tree provides a canonical labelling of its leaves, in the following way. Assuming that the plane is oriented, the leaves are labelled from 1 to  $n$ , from left to right, the root being at the top and the leaves at the bottom.

There exists a bijection between the set of dyadic subdivisions of  $[0, 1]$  and the set of finite binary rooted planar trees. Indeed, given such a tree, one may label its vertices by dyadic intervals in the following way. First, the root is labelled by  $[0, 1]$ . Suppose that a vertex is labelled by  $I = [\frac{k}{2^n}, \frac{k+1}{2^n}]$ , then its two descendant vertices are labelled by the two halves  $I$ :  $[\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}]$  for the left one and  $[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}]$  for the right one. Finally, the dyadic subdivision associated to the tree is the sequence of intervals which label its leaves.

Thus, an element  $h$  of  $V$  is represented by a triple  $(\tau_1, \tau_0, \sigma)$ , where  $\tau_0$  and  $\tau_1$  have the same number of leaves  $n \in \mathbb{N}^*$ , and  $\sigma \in \mathfrak{S}_n$ . Such a triple will be called a *symbol* for  $h$ . It is convenient to interpret the permutation  $\sigma$  as the bijection  $\varphi_\sigma$  which maps the  $i$ -th leaf of the source tree  $\tau_0$  to the  $\sigma(i)$ -th leaf of the target tree  $\tau_1$ . When  $h$  belongs to  $F$ , the permutation  $\sigma$  is identity and the symbol reduces to a pair of trees  $(\tau_1, \tau_0)$ .

Now, two symbols are equivalent if they represent the same element of  $V$  and one denotes by  $[\tau_1, \tau_0, \sigma]$  the equivalence class. The composition law of piecewise dyadic affine bijections is pushed out on the set of equivalence classes of symbols in the following way. In order to define  $[\tau'_1, \tau'_0, \sigma'] \cdot [\tau_1, \tau_0, \sigma]$ , one may suppose, at the price of refining both symbols, that the tree  $\tau_1$  coincides with the tree  $\tau'_0$ . Then the product of the two symbols is

$$[\tau'_1, \tau_1, \sigma'] \cdot [\tau_1, \tau_0, \sigma] = [\tau'_1, \tau_0, \sigma' \circ \sigma].$$

It follows that  $V$  is isomorphic to the group of equivalence classes of symbols endowed with this internal law.

## 2.3 Partial automorphisms of trees

The beginning of the article [20] formalizes a change of point of view, consisting in considering, not the finite binary trees, but their complements in the infinite binary tree.

Let  $\mathcal{T}_2$  be the infinite binary rooted planar tree (all its vertices other than the root are 3-valent). Each finite binary rooted planar tree  $\tau$  can be embedded in a unique way into  $\mathcal{T}_2$ , assuming that the embedding maps the root of  $\tau$  onto the root of  $\mathcal{T}_2$ , and respects the orientation. Therefore,  $\tau$  may be identified with a subtree of  $\mathcal{T}_2$ , whose root coincides with that of  $\mathcal{T}_2$ .

**Definition 7** (cf. [24]). A partial isomorphism of  $\mathcal{T}_2$  consists of the data of two finite binary rooted subtrees  $\tau_0$  and  $\tau_1$  of  $\mathcal{T}_2$  having the same number of leaves  $n \in \mathbb{N}^*$ , and an isomorphism  $q : \mathcal{T}_2 \setminus \tau_0 \rightarrow \mathcal{T}_2 \setminus \tau_1$ . The complements of  $\tau_0$  and  $\tau_1$  have  $n$  components, each one isomorphic to  $\mathcal{T}_2$ , which are enumerated from 1 to  $n$  according to the labelling of the leaves of the trees  $\tau_0$  and  $\tau_1$ . Thus,  $\mathcal{T}_2 \setminus \tau_0 = T_0^1 \cup \dots \cup T_0^n$  and  $\mathcal{T}_2 \setminus \tau_1 = T_1^1 \cup \dots \cup T_1^n$  where the  $T_j^i$ 's are the connected components. Equivalently, the partial isomorphism of  $\mathcal{T}_2$  is given by a permutation  $\sigma \in \mathfrak{S}_n$  and, for  $i = 1, \dots, n$ , an isomorphism  $q_i : T_0^i \rightarrow T_1^{\sigma(i)}$ .

Two partial automorphisms  $q$  and  $r$  can be composed if and only if the target of  $r$  coincides with the source of  $q$ . One gets the partial automorphism  $q \circ r$ . The composition provides a structure of inverse monoid on the set of partial automorphisms.

Let  $\partial\mathcal{T}_2$  be the boundary of  $\mathcal{T}_2$  (also called the set of “ends” of  $\mathcal{T}_2$ ) endowed with its usual topology, for which it is a Cantor set. Although a partial automorphism does not act (globally) on the tree, it does act on its

boundary. One has therefore a morphism from the monoid of partial isomorphism into the homeomorphisms of  $\partial\mathcal{T}_2$ , whose image  $N$  is the *spheromorphism group of Neretin* (see [28]).

Thompson's group  $V$  can be viewed as the subgroup of  $N$  which is the image of those partial automorphisms which respect the local orientation of the edges.

## 2.4 Generalizations following Brown and Stein, Bieri and Strebel

Brown considered in [9] similar groups  $F_{n,r} \subset T_{n,r} \subset V_{n,r}$ , extending previous work of Higman, which were defined as in the last two constructions above but using instead of binary trees forests of  $r$  copies of  $n$ -ary trees so that  $F, T, V$  correspond to  $n = 2$  and  $r = 1$ . The isomorphism type of  $V_{n,r}$  and  $T_{n,r}$  only depends on  $r \pmod n$  while  $F_{n,r}$  depends only on  $n$ . We drop the subscript  $r$  when  $r = 1$ . These groups are finitely presented and of type  $FP_\infty$  according to [8] for the case of  $F$  and  $T$  and then ([9], thm. 4.17) for its extension to all other groups from this family. Moreover, Higman have proved (see [22]) that  $V_{n,r}$  has a simple subgroup of index  $\text{g.c.d}(2, n-1)$ , and this was extended by Brown who showed that  $F_n$  have simple commutator and  $T_{n,r}$  have simple double commutator groups (see [9] for more details and refinements).

One can obtain these groups also by considering  $n$ -adic piecewise affine homeomorphisms (or bijections) of  $[0, r]$  (with identified endpoints for  $T_{n,r}$ ) i.e. having singularities in  $\mathbb{Z}[\frac{1}{n}]$  and derivatives in  $\{n^a, a \in \mathbb{Z}\}$ . This point of view was taken further by Bieri, Strebel and Stein in [2, 33]. Specifically, given a multiplicative subgroup  $P \subset \mathbb{R}$ , a  $\mathbb{Z}[P]$ -submodule  $A \subset \mathbb{R}$  satisfying  $P \cdot A = A$ , and a positive  $r \in A$ , one can consider the group  $F_{A,P,r}$  of those PL homeomorphisms of  $[0, r]$  with finite singular set in  $A$  and all slopes in  $P$ . There are similar families  $T_{A,P,r}$  and  $V_{A,P,r}$ . Brown and Stein proved that  $F_{\mathbb{Z}[\frac{1}{n_1 n_2 \dots n_k}], \langle n_1, n_2, \dots, n_k \rangle, r}$  is finitely presented of  $FP_\infty$  type. Furthermore  $F_{A,P,r}$  and  $V_{A,P,r}$  have simple commutator subgroups, while  $T_{A,P,r}$  have simple second commutator subgroup.

There is a natural extension  $V_{n,r}^\pm$  of  $V_{n,r}$  by the sum  $\oplus_0^\infty \mathbb{Z}/2\mathbb{Z}$ , when we consider both orientation preserving and orientation reversing piecewise affine homeomorphisms.

## 2.5 Mapping class groups of infinite surfaces and braided Thompson groups

Let  $\mathcal{S}_{0,\infty}$  be the oriented surface of genus zero, which is the following inductive limit of compact oriented genus zero surfaces with boundary  $\mathcal{S}_n$ . Starting with a cylinder  $\mathcal{S}_1$ , one gets  $\mathcal{S}_{n+1}$  from  $\mathcal{S}_n$  by gluing a pair of pants (i.e. a three-holed sphere) along each boundary circle of  $\mathcal{S}_n$ . This construction yields, for each  $n \geq 1$ , an embedding  $\mathcal{S}_n \hookrightarrow \mathcal{S}_{n+1}$ , with an orientation on  $\mathcal{S}_{n+1}$  compatible with that of  $\mathcal{S}_n$ . The resulting inductive limit (in the topological category) of the  $\mathcal{S}_n$ 's is the surface  $\mathcal{S}_{0,\infty} = \lim_{\substack{\longrightarrow \\ n}} \mathcal{S}_n$ .

By the above construction, the surface  $\mathcal{S}_{0,\infty}$  is the union of a cylinder and of countably many pairs of pants. This topological decomposition of  $\mathcal{S}_{0,\infty}$  will be called the *canonical pair of pants decomposition*.

The set of isotopy classes of orientation-preserving homeomorphisms of  $\mathcal{S}_{0,\infty}$  is an *uncountable* group. By restricting to a certain type of homeomorphisms (called asymptotically rigid), we shall obtain countable subgroups (see [17, 18]).

Any connected and compact subsurface of  $\mathcal{S}_{0,\infty}$  which is the union of the cylinder and finitely many pairs of pants of the canonical decomposition will be called an *admissible subsurface* of  $\mathcal{S}_{0,\infty}$ . The *type* of such a subsurface  $S$  is the number of connected components in its boundary.

**Definition 8** (following [24, 17]). *A homeomorphism  $\varphi$  of  $\mathcal{S}_{0,\infty}$  is asymptotically rigid if there exist two admissible subsurfaces  $S_0$  and  $S_1$  having the same type, such that  $\varphi(S_0) = S_1$  and whose restriction  $\mathcal{S}_{0,\infty} \setminus S_0 \rightarrow \mathcal{S}_{0,\infty} \setminus S_1$  is rigid, meaning that it maps each pants (of the canonical pants decomposition) onto a pants.*

*The asymptotically rigid mapping class group of  $\mathcal{S}_{0,\infty}$  is the group of isotopy classes of asymptotically rigid homeomorphisms.*

The *asymptotically rigid mapping class group* of  $\mathcal{S}_{0,\infty}$  is a finitely presented group  $\mathcal{B}$  (see [17]) which fits into the exact sequence:

$$1 \rightarrow PM(\mathcal{S}_{0,\infty}) \rightarrow \mathcal{B} \rightarrow V \rightarrow 1.$$



Some very similar versions of the same group (using a Cantor disk instead of a Cantor sphere or a more combinatorial framework) were obtained independently by Brin ([6]) and Dehornoy ([12]). We will call any version of them as *braided Thompson groups*.

## 2.6 Brin's groups $nV$

A rather different direction was taken in the seminal paper [5] of Brin, where the author constructed a family of countable groups  $nV$  acting as homeomorphisms of the product of  $n$ -copies of the standard triadic Cantor, generalizing the group  $V$  which occurs for  $n = 1$ .

Let  $I^n \subset \mathbb{R}^n$  denote the unit cube. A *numbered pattern* is a finite dyadic partition of  $I^n$  into parallelepipeds along with a numbering. A dyadic partition is obtained from the cube by dividing at each step of the process one parallelepiped into two equal halves by a cutting hyperplane parallel to one of the coordinates hyperplane.

One definition of  $nV$  is as the group of piecewise affine (not continuous!) transformations associated to pairs of numbered patterns. Given the numbered patterns  $P = (L_1, L_2, \dots, L_n)$  and  $Q = (R_1, R_2, \dots, R_n)$ , we set  $\varphi_{P,Q}$  for the unique piecewise affine transformation of the cube sending affinely each  $L_i$  into  $R_i$  and preserving the coordinates hyperplanes. Thus  $nV$  is the group of piecewise affine transformations of the form  $\varphi_{P,Q}$ , with  $P, Q$  running over the set of all possible dyadic partitions.

Another description is as a group of homeomorphisms of the product  $C^n$  of the standard triadic Cantor set  $C$ . Parallelepipeds in a dyadic partition correspond to closed and open (clopen) subsets of  $C^n$ . Every dyadic cutting hyperplane  $H$  subdividing some parallelepiped  $R$  into two halves determines a parallel shadow (open) parallelepiped in  $R$  whose width is one third of the width of  $R$  in the direction orthogonal to  $H$ . Notice then that the complement of the union of all shadow parallelepipeds is  $C^n$ . Every pattern  $P = (R_1, R_2, \dots, R_n)$  determines a numbered collection of parallelepipeds  $X_P = (X(R_1), X(R_2), \dots, X(R_n))$  whose complementary is the set of shadows parallelepipeds of those cutting hyperplanes used to built  $P$ . Then  $A(R_i) = X(R_i) \cap C^n$  form a clopen partition of  $C^n$ . For a pair of patterns  $P, Q$  we define the homeomorphism  $h_{P,Q}$  of  $C^n$  as the unique homeomorphism which sends affinely  $A(L_i)$  into  $A(R_i)$  and preserves the orientation in each coordinate. This amounts to say that  $h_{P,Q}$  is the restriction to  $C^n$  of the piecewise affine transformation sending affinely  $X(L_i)$  into  $X(R_i)$  and preserving the coordinates hyperplanes.

The groups  $nV$  are simple (see e.g. [5, 7]) and finitely presented (see [21]). The stabilizer at some  $a \in C^n$  of the (germs of) homeomorphisms in  $nV$  is isomorphic to  $\mathbb{Z}^{r(a)}$ , where  $r(a)$  is the number of rational coordinates of  $a$ . This implies that the groups  $nV$  are pairwise non-isomorphic (see [4] for details).

We could of course extend this construction to arbitrary products of central Cantor sets  $C_\lambda$  in the spirit of Brown and Stein, Bieri and Strebel as above.

As in the case of groups  $V_{n,r}$  there exists an extension  $nV^{sym}$  of  $nV$  by the sum  $\oplus_0^\infty D_n$ , where  $D_n$  is the group of orientation preserving symmetries of the cube. Its elements correspond to numbered patterns  $P = (L_1, L_2, \dots, L_n)$  and  $Q = (R_1, R_2, \dots, R_n)$ , along with a  $n$ -tuple  $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$  of orientation preserving symmetries of the  $n$ -cube. The map  $\varphi_{P,Q,\Sigma}$  consists of the unique piecewise affine transformation sending  $\sigma_i(L_i)$  into  $R_i$ . Recall that  $D_n$  is the group of orthogonal  $n \times n$  matrices with integer entries and unit determinant.

## 3 Proof of general countability statements

### 3.1 Proof of Theorem 1

We parameterize the interval  $E$  containing the Cantor set  $C$  by the  $\mathcal{C}^k$ -curve  $\gamma : [0, 1] \rightarrow M$  and denote by  $A \subset [0, 1]$  the preimage of  $C$ , which is still a Cantor set. We may assume that  $\{0, 1\} \subset A$ . For the sake of simplicity we suppose that the interval  $E$  lies in the interior of  $M$ . The proof works in general, with only minor modifications. Let  $\varphi \in \text{Diff}^{k,+}(M, C)$  and denote by  $\xi(t) = \varphi \circ \gamma(t)$ . Consider a  $\mathcal{C}^k$ -coordinates chart  $U \subset M$  containing  $E$ , such that  $U$  is identified with an open disk, while  $\gamma$  is now linear and parameterized by arc length, namely that  $\|\dot{\gamma}\| = 1$  and  $\ddot{\gamma} = 0$ . The norm  $\|\cdot\|$  is associated to the standard scalar product  $\langle, \rangle$  on  $U$  induced from  $\mathbb{R}^n$ .

The strategy of the proof is as follows. We define a subset  $I_\varepsilon \subset [0, 1]$  consisting of finitely many intervals which contains  $A$ . At first one straightens out  $\xi$  in the complementary of  $I_\varepsilon$ . To this purpose we modify  $\varphi$  by composing with a convenient compactly supported diffeomorphism. Further we show that there is an isotopy  $\text{rel } A$  which straightens out the remaining arcs of  $\xi$ . Eventually, one proves that a diffeomorphism preserving the orientation of the surface which fixes the arc  $E$  is, up to isotopy, supported outside a disk neighborhood of  $E$ . This will show that  $\varphi$  has a compactly supported class.

Assume for the moment that  $A$  is just an infinite set without isolated points. The set of those  $t$  for which  $\gamma(t) = \xi(t)$  is a closed subset of  $[0, 1]$  containing  $A$  and hence its closure  $\overline{A}$ . Let now  $t_0 \in \overline{A}$ . Then, since  $\gamma$  and  $\xi$  are differentiable at  $t_0$  we have:

$$\dot{\gamma}(t_0) = \lim_{t \in A, t \rightarrow t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \lim_{t \in A, t \rightarrow t_0} \frac{\xi(t) - \xi(t_0)}{t - t_0} = \dot{\xi}(t_0). \quad (6)$$

If  $\varphi$  is twice differentiable then the same argument shows that:

$$\ddot{\gamma}(t_0) = \ddot{\xi}(t_0). \quad (7)$$

Since  $\varphi$  is of class  $\mathcal{C}^2$ , for every  $\varepsilon > 0$ , there exists  $\delta(\varepsilon) > 0$  such that whenever  $s_1, s_2 \in A$ , with  $|s_1 - s_2| < \delta(\varepsilon)$  we have:

$$1 - \varepsilon < \langle \dot{\gamma}(t), \dot{\xi}(t) \rangle \leq 1, \text{ for all } t \in [s_1, s_2], \quad (8)$$

$$|\ddot{\xi}(t)| < \varepsilon, \text{ for all } t \in [s_1, s_2]. \quad (9)$$

We assume now that  $A = \cap_{j=1}^\infty A_j$  is the infinite nested intersection of the closed finite unions of intervals  $A_j \supset A_{j+1} \supset \dots$ .

We denote  $I_\varepsilon = \cup_{s_1, s_2 \in A; |s_1 - s_2| \leq \frac{\delta(\varepsilon)}{2}} [s_1, s_2] \subset [0, 1]$ . We choose  $\varepsilon > 0$  small enough such that the image of  $\xi|_{I_\varepsilon}$  is contained within the coordinates disk  $U$ .

Set further  $\gamma_s(t) = (1 - s)\gamma(t) + s\xi(t)$ , for  $t \in [s_1, s_2] \subset I_\varepsilon$  and  $s \in [0, 1]$ .

**Lemma 1.** *Fix  $\varepsilon < 1$  as above. Let  $s_1, s_2 \in A$ , such that  $|s_1 - s_2| \leq \delta(\varepsilon)/2$ . Then  $\gamma_s|_{[s_1, s_2]}$  provides a  $\mathcal{C}^k$ -isotopy between the restrictions  $\gamma|_{[s_1, s_2]}$  and  $\xi|_{[s_1, s_2]}$  to the interval  $[s_1, s_2]$ . In particular, the image  $\xi(I_\varepsilon)$  is contained within the union of orthogonal strips  $(\gamma(I_\varepsilon) \times \mathbb{R}) \cap U$ .*

*Proof.* We have to prove that for any  $s \in [0, 1]$  the curve  $\gamma_s|_{[s_1, s_2]}$  is simple. This follow immediately from the fact that whenever  $\varepsilon < 1$  we have:

$$\langle \dot{\gamma}_s(t), \dot{\gamma}(t) \rangle \geq 1 - s + s\langle \dot{\xi}(t), \dot{\gamma}(t) \rangle \geq 1 - \varepsilon s > 0 \quad (10)$$

for any  $t \in [0, 1]$ ,  $s \in [0, 1]$ . Further, note that the curve  $\gamma_s(t)$ , for  $s \in [0, 1]$  and fixed  $t \in I_\varepsilon$  is a segment joining  $\xi(t)$  with its orthogonal projection onto  $\gamma(I_\varepsilon)$ .  $\square$

We set

$$\eta(t) = \begin{cases} \xi(t), & \text{if } t \in I_\varepsilon; \\ \gamma(t), & \text{if } t \notin I_\varepsilon. \end{cases} \quad (11)$$

**Lemma 2.** *There exists a compactly supported diffeomorphism  $\psi \in \text{PDiff}^{k,+}(M, C)$  such that  $\psi(\xi)$  and  $\eta$  are isotopic  $\text{rel } A$ .*

*Proof.* Lemma 1 shows that the image of  $\eta$  is a simple curve, as  $\xi(I_\varepsilon)$  is contained within the union of orthogonal strips  $(\gamma(I_\varepsilon) \times \mathbb{R}) \cap U$ , and thus it cannot intersect  $\gamma([0, 1] \setminus I_\varepsilon)$ .

Note that  $A \subset I_\varepsilon$ , since  $A$  has no isolated points. The endpoints of a maximal complementary interval should belong to  $A$ , by maximality. In particular, its length should be greater than  $\frac{\delta(\varepsilon)}{2}$ , and hence there are only finitely many maximal complementary intervals say  $J_1, J_2, \dots, J_p$ . Then  $\xi(J_i)$  are pairwise disjoint smooth arcs whose interiors are  $\xi(\text{int}(J_i)) \subset M - C$ , each such arc joining two distinct points of  $C$ . Moreover, as  $\xi(t) = \gamma(t)$ , for  $t \in \cup_{i=1}^p \partial J_i$ , we can straighten out the half-arcs of  $\xi$  around these points. Namely, there

exists a small neighborhood  $N(I_\varepsilon)$  of  $I_\varepsilon$  within  $[0, 1]$  such that after perturbing  $\xi$  by an isotopy supported in  $N(I_\varepsilon) \cap (\cup_{i=1}^p J_i)$  we have  $\xi(t) = \gamma(t)$ , for  $t \in N(I_\varepsilon) \cap (\cup_{i=1}^p J_i)$ .

Now the arcs  $\xi(I_\varepsilon)$  are disjoint both from  $\xi(J_i \setminus N(I_\varepsilon))$  and  $\gamma(J_i \setminus N(I_\varepsilon))$ . There exists then a small enough open neighborhood  $V$  of  $\xi(I_\varepsilon)$  within  $U$  which is disjoint from both  $\xi(J_i \setminus N(I_\varepsilon))$  and  $\gamma(J_i \setminus N(I_\varepsilon))$ . Therefore there exists an orientation preserving diffeomorphism  $\psi$  supported on  $M - V$ , thus compactly supported, such that  $\psi(\xi(J_i \setminus N(I_\varepsilon))) = \gamma(J_i \setminus N(I_\varepsilon))$ , and hence  $\psi(\xi(J_i)) = \eta(J_i)$ . Thus  $\psi(\xi)$  and  $\eta$  are isotopic rel  $A$ , as claimed.  $\square$

**Lemma 3.** *The curves  $\gamma$  and  $\eta$  are isotopic rel  $A$ .*

*Proof.* We will prove that the family

$$\eta_s(t) = \begin{cases} \gamma_s(t), & \text{if } t \in I_\varepsilon; \\ \gamma(t), & \text{if } t \notin I_\varepsilon \end{cases} \quad (12)$$

is the desired isotopy. From Lemma 1 it suffices to show that there are not intersections between the segments of curves  $\gamma_s|_{[s_1, s_2]}$  and  $\gamma_s|_{[s_3, s_4]}$ , when  $s_i \in A$  and  $[s_1, s_2], [s_3, s_4] \subset I_\varepsilon$  are disjoint.

Let  $p = \gamma_s|_{[s_1, s_2]}(t_0)$  be a point on the first curve segment. We want to estimate the angle  $\beta$  of the Euclidean triangle with vertices  $p, \gamma(s_1), \gamma(s_2)$  at  $\gamma_s(s_1)$ . We can write then:

$$\langle \gamma_s(t_0) - \gamma_s(s_1), \dot{\gamma}(0) \rangle = \int_0^{t_0 - s_1} \langle \dot{\gamma}_s(s_1 + x), \dot{\gamma}(0) \rangle dx = \int_0^{t_0 - s_1} 1 - s + s \langle \dot{\xi}(s_1 + x), \dot{\gamma}(0) \rangle dx. \quad (13)$$

Then (8) implies:

$$\| \gamma_s(t_0) - \gamma_s(s_1) \| \cos(\beta) = \langle \gamma_s(t_0) - \gamma_s(s_1), \dot{\gamma}(0) \rangle \geq (t_0 - s_1)(1 - s\varepsilon). \quad (14)$$

On the other hand from (9) we derive

$$\| \dot{\xi}(x) - \dot{\xi}(s_1) \| \leq \varepsilon(x - s_1), \quad (15)$$

and then:

$$\| \gamma_s(t_0) - \gamma_s(s_1) \| \leq \int_0^{t_0 - s_1} \| \dot{\gamma}_s(x) \| dx \leq \int_0^{t_0 - s_1} (s \| \dot{\xi}(x) \| + (1 - s)) dx \leq t_0 - s_1 + \frac{\varepsilon}{2}(t_0 - s_1)^2. \quad (16)$$

From (14) we obtain

$$\cos(\beta) \geq \frac{1 - s\varepsilon}{1 + \frac{\varepsilon}{2}(t_0 + s_1)} \geq \frac{1 - \varepsilon}{1 + \varepsilon}. \quad (17)$$

If we choose  $\varepsilon \leq \frac{1}{3}$  then  $\beta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ .

Assume now the contrary of our claim, namely that there exists some intersection point  $p$  between  $\gamma_s|_{[s_1, s_2]}$  and  $\gamma_s|_{[s_3, s_4]}$ . Up to a symmetry of indices we can assume that the Euclidean triangle with vertices at  $p, \gamma_s(s_1)$  and  $\gamma_s(s_2)$  has the angle  $\beta$  at  $\gamma_s(s_1)$  within the interval  $[\frac{\pi}{2}, \pi)$ . This contradicts our estimates (17) for  $\beta$ .  $\square$

The last ingredient of the proof of Theorem 1 is the following:

**Lemma 4.** *Assume that there exists an isotopy of class  $\mathcal{C}^k$  between  $\gamma$  and  $\eta = \psi(\varphi(\gamma))$  rel  $A$ . Then  $\varphi$  is  $\mathcal{C}^k$ -isotopic to a compactly supported diffeomorphism from  $\text{PDiff}^{k,+}(M, C)$ .*

*Proof.* We can assume that the disk  $D$  of diameter  $E$  is contained in  $U$ .

If the dimension of  $M$  is 2, the endpoints of  $E$  separate the circle  $\partial D$  into two arcs, say  $F^+$  and  $F^-$ . The circular order of the three arcs  $F^+, E, F^-$  around an endpoint of  $E$  is preserved by  $\psi \circ \varphi \in \text{Diff}^{k,+}(M, C)$ . Note that  $E$  is fixed by  $\psi \circ \varphi$ . Thus there exists a  $\mathcal{C}^k$ -isotopy which is identity on  $E$ , sends  $\psi(\varphi(F^+))$  to  $F^+$  and  $\psi(\varphi(F^-))$  to  $F^-$ . Therefore  $\psi \circ \varphi$  is isotopic to a diffeomorphism supported on the complement of  $D$  and hence its class is compactly supported. The claim follows now, because  $\psi$  is equally compactly supported.

If the dimension of  $M$  is at least 3, there exists an isotopy of  $M$  sending  $\psi(\varphi(\partial D))$  to  $\partial D$ , which is identity on  $E$ , because  $\psi \circ \varphi \in \text{PDiff}^{k,+}(M, C)$  and we conclude as above.  $\square$

## 3.2 Sparse sets and proofs of Theorems 2, 3 and 4

### 3.2.1 Preliminaries

Let  $\mathcal{N}_\varepsilon(a)$  denote the  $\varepsilon$ -neighborhood  $|x - a| < \varepsilon$  of  $a$  in  $\mathbb{R}$ ,  $\mathcal{N}_\varepsilon^\pm(a)$  the punctured right and left semi-neighborhoods of  $a$ , i.e.,  $a < x < a + \varepsilon$  and  $a - \varepsilon < x < a$ , respectively.

We say that  $a$  is a *left point* of  $C$  if there is a left semi-neighborhood  $\mathcal{N}^-(a)$  such that  $\mathcal{N}^-(a) \cap C = \emptyset$ . In the same way we define *right points*.

For  $a \in C$  denote by  $\text{Diff}_a^k$  the stabilizer of  $a$  in  $\text{Diff}^k(\mathbb{R}, C)$ , and by  $\mathfrak{diff}_a^k$  the group of  $k$ -germs of elements of the stabilizer of  $a$  in  $\mathfrak{diff}^k(C)$ . The superscript  $+$  in  $\text{Diff}_a^{k,+}$  and  $\mathfrak{diff}_a^{k,+}$  means that we only consider those diffeomorphisms that preserve the orientation of the interval, i.e. increasing.

Let  $\varphi$  be a diffeomorphism with  $\varphi(a) = a$ . We say that  $\varphi$  is *N-flat* at  $a$  if:

$$\varphi(x) - x = o((x - a)^N), \quad \text{as } x \rightarrow a. \quad (18)$$

**Lemma 5.** *Assume that  $C$  is a  $\sigma$ -sparse subset of  $\mathbb{R}$ . Let  $\varphi \in \text{Diff}_a^1$  be 1-flat at  $a \in C$ . Then  $\varphi|_C$  is the identity in a small neighborhood of  $a$ .*

*Proof.* Observe first that  $\varphi \in \text{Diff}_a^{1,+}$ , since  $\varphi'(a) = 1$  and hence  $\varphi$  must be increasing. We can assume without loss of generality that  $a$  is not a right point of  $C$ . Suppose that  $\varphi$  is nontrivial on  $\mathcal{N}_\delta^+(a) \cap C$  for any  $\delta > 0$ .

We first claim that fixed points of  $\varphi$  accumulate from the right to  $a$ . Otherwise, there exists some  $\delta$  such that  $\varphi(x) - x$  keeps constant sign for all  $x \in \mathcal{N}_\delta^+(a)$ . Assume that this sign is positive and choose  $b \in \mathcal{N}_\delta^+(a) \cap C$ . Let  $(\alpha, \beta) \subset (a, b)$  be a maximal complementary interval of length at least  $\sigma(b - a)$ . By maximality  $\alpha \in C$ . Since  $\varphi(\alpha) \in C$  and  $\varphi(\alpha) > \alpha$  we have  $\varphi(\alpha) \geq \beta$ , so that:

$$\frac{\varphi(\alpha) - a}{\alpha - a} = \frac{\varphi(\alpha) - \alpha}{\alpha - a} + 1 \geq \frac{\beta - \alpha}{\alpha - a} + 1 \geq \frac{\sigma(b - a)}{\alpha - a} + 1 \geq 1 + \sigma. \quad (19)$$

By the mean value theorem there exists  $\xi \in (a, \alpha)$  such that:

$$\varphi'(\xi) = \frac{\varphi(\alpha) - a}{\alpha - a} \geq 1 + \sigma.$$

But this inequality contradicts the 1-flatness condition for small  $\delta$ , as taking the limit when  $\delta \rightarrow 0$  we would obtain  $\varphi'(a) \geq 1 + \sigma$ .

When the sign of  $\varphi(x) - x$  is negative we reach the same conclusion by considering  $\varphi(\beta) - \beta$ . This proves the claim.

Therefore there is a decreasing sequence  $u_k$  accumulating at  $a$ , such that  $\varphi(u_k) = u_k$ . As  $\varphi|_{C \cap \mathcal{N}_\delta^+(a)}$  is not identity for any  $\delta > 0$  there exists a decreasing sequence  $v_k \in C$  accumulating on  $a$ , such that all  $\varphi(v_k) - v_k$  are of the same sign, say positive. Therefore, up to passing to a subsequence, we obtain a sequence of disjoint intervals  $(\alpha_j, \beta_j)$  such that  $\beta_{j+1} \leq \alpha_j$ ,  $\varphi(\alpha_j) = \alpha_j$ ,  $\varphi(\beta_j) = \beta_j$ , and  $v_j \in (\alpha_j, \beta_j)$ .

Since  $\varphi$  is monotone, it has to be monotone increasing, by above. Thus  $\varphi^k(v_j) \in [\alpha_j, \beta_j]$ , for any  $k \in \mathbb{Z}$ , where  $\varphi^k$  denotes the  $k$ -th iterate of  $\varphi$ . The bi-infinite sequence  $\varphi^k(v_j)$  is increasing and so:

$$\alpha_j \leq \lim_{k \rightarrow -\infty} \varphi^k(v_j) < \lim_{k \rightarrow \infty} \varphi^k(v_j) \leq \beta_j. \quad (20)$$

Now  $\lim_{k \rightarrow -\infty} \varphi^k(v_j)$  and  $\lim_{k \rightarrow \infty} \varphi^k(v_j)$  are fixed points of  $\varphi$  and we can assume, without loss of generality that our choice of intervals is such that  $\alpha_j = \lim_{k \rightarrow -\infty} \varphi^k(v_j)$ ,  $\lim_{k \rightarrow \infty} \varphi^k(v_j) = \beta_j$ . In particular  $\alpha_j, \beta_j \in C$ .

As  $C$  is  $\sigma$ -sparse there is a complementary interval  $(\gamma_j, \delta_j) \subset (\alpha_j, \beta_j)$  of length at least  $\sigma(\beta_j - \alpha_j)$ . The interval  $(\gamma_j, \delta_j)$  cannot contain any point  $\varphi^k(v_j)$  and thus there exists some  $k_j \in \mathbb{Z}$  such that

$$\varphi^{k_j}(v_j) \leq \gamma_j < \delta_j \leq \varphi^{k_j+1}(v_j). \quad (21)$$

Denote  $\varphi^{k_j}(v_j) = \eta_j$ . We have then

$$\frac{\varphi(\eta_j) - \varphi(\alpha_j)}{\eta_j - \alpha_j} - 1 = \frac{\varphi(\eta_j) - \eta_j}{\eta_j - \alpha_j} \geq \frac{\sigma(\beta_j - \alpha_j)}{\eta_j - \alpha_j} \geq \sigma. \quad (22)$$

By the mean value theorem there exists  $\xi_j \in (\alpha_j, \eta_j)$  such that

$$\frac{\varphi(\eta_j) - \varphi(\alpha_j)}{\eta_j - \alpha_j} = \varphi'(\xi_j). \quad (23)$$

and thus such that  $\varphi'(\xi_j) \geq 1 + \sigma$ . As  $\varphi'$  is continuous at  $a$ , by letting  $j$  go to infinity we derive  $\varphi'(a) \geq 1 + \sigma$  which contradicts the 1-flatness.  $\square$

**Lemma 6.** *If  $C$  is  $\sigma$ -sparse and  $\varphi \in \text{Diff}_a^1$  is not 1-flat then*

$$|\varphi'(a) - 1| \geq \sigma. \quad (24)$$

*Proof.* Let  $\varphi \in \text{Diff}_a^1$  not 1-flat, so that  $\varphi'(a) \neq 1$ . Let us further suppose that  $\varphi'(a) > 1$ , the other situation being similar. For any  $\delta > 0$  we can choose  $b \in \mathcal{N}_\delta^+(a) \cap C$ . There is then a maximal complementary interval  $(\alpha, \beta) \subset (a, b)$  of length at least  $\sigma(b - a)$ . By maximality  $\alpha \in C$ .

We claim that for small enough  $\delta$  we have  $\varphi(\alpha) > \alpha$ . Assume the contrary. By the mean value theorem there exists  $\xi \in (a, \alpha) \subset (a, b)$  such that

$$\varphi'(\xi) = 1 + \frac{\varphi(\alpha) - \alpha}{\alpha - a} \leq 1 \quad (25)$$

and letting  $\delta$  go to 0 we would obtain  $\varphi'(a) \leq 1$ , contradicting our assumptions. Thus  $\varphi(\alpha) > \alpha$ , and hence  $\varphi(\alpha) \geq \beta$ . As above, the mean value theorem provides us  $\xi \in (a, \alpha)$  so that

$$\varphi'(\xi) = 1 + \frac{\varphi(\alpha) - \alpha}{\alpha - a} \geq 1 + \sigma. \quad (26)$$

Letting  $\delta$  go to zero we obtain  $\varphi'(a) \geq 1 + \sigma$ . When  $\varphi'(a) < 1$  we can use similar methods or pass to  $\varphi^{-1}$  in order to obtain  $\varphi'(a) \leq 1 - \sigma$ .  $\square$

**Lemma 7.** *Let  $C$  sparse and  $a \in C$ . Then one of the following holds:*

1. *either for any  $\varphi \in \text{Diff}_a^{1,+}$ , the restriction  $\varphi|_C$  is identity in a small neighborhood of  $a$ , so that  $\mathfrak{diff}_a^{1,+} = 1$ ;*
2. *or else, there is  $\psi_a \in \text{Diff}_a^{1,+}$  such that for any  $\varphi \in \text{Diff}_a^{1,+}$  the restriction of  $\varphi$  to a small neighborhood  $\mathcal{N}_\delta(a) \cap C$  coincides with the iterate  $\psi_a^k|_C$  for some  $k \in \mathbb{Z} \setminus \{0\}$ . Moreover, any such  $\psi_a$  is of the form:*

$$\psi_a(x) = a + p(x - a) + o(x - a), \quad \text{as } x \rightarrow a, \quad (27)$$

*where  $|p - 1| \geq \sigma$ . Thus  $\mathfrak{diff}_a^{1,+} = \mathbb{Z}$ .*

*Proof.* If the first alternative doesn't hold, by Lemma 5 we can assume that there exists some  $\varphi \in \text{Diff}_a^{1,+}$  which is not 1-flat.

The map  $\chi : \text{Diff}_a^1 \rightarrow \mathbb{R}^*$  given by  $\chi(\varphi) = \varphi'(a)$  is easily seen to be a group homomorphism. By Lemma 6 the subgroup  $\chi(\text{Diff}_a^{1,+})$  of  $\mathbb{R}_+^*$  is discrete and non-trivial and thus it is isomorphic to  $\mathbb{Z}$ . Let  $\psi_a \in \text{Diff}_a^{1,+}$  be a germ whose image  $\chi(\psi_a)$  is a generator of  $\chi(\text{Diff}_a^{1,+})$ . Then  $\psi_a$  is not 1-flat and thus, by Lemma 6, it satisfies the equation (27).

If  $\varphi \in \text{Diff}_a^{1,+}$ , then we can write  $\varphi = \psi_a^k \theta$ , for some  $k \in \mathbb{Z} \setminus \{0\}$  and  $\theta \in \ker \chi$ . But the kernel of  $\chi$  consists of those  $\theta \in \text{Diff}_a^{1,+}$  which are 1-flat. By Lemma 5 the restriction of  $\theta$  to some neighborhood  $\mathcal{N}_\delta(a) \cap C$  is identity. This proves that  $\varphi = \psi_a^k$  in a neighborhood  $\mathcal{N}_\delta(a) \cap C$ , as claimed.  $\square$



**Remark 3.** If  $C = C_\lambda$  is the ternary central Cantor set in  $\mathbb{R}$ , then  $\mathfrak{diff}_a^{1,+}(C_\lambda)$  is not always  $\mathbb{Z}$ . An element  $a$  of  $C_\lambda$  is called  $\lambda$ -rational if it has an eventually periodic development

$$a = \sum_{i=1}^{\infty} a_i \lambda^i,$$

where  $a_i \in \{0, \lambda - 1\}$ . Therefore  $\mathfrak{diff}_a^{1,+}(C_\lambda)$  is  $\mathbb{Z}$  if and only if  $a$  is  $\lambda$ -rational and trivial, otherwise.

**Remark 4.** Since the subgroup  $\chi(\text{Diff}_a^1) \subset \mathbb{R}^*$  is discrete there exists  $\lambda \geq 1$  such that  $\chi(\text{Diff}_a^1)$  is of the form  $\langle \lambda \rangle$ ,  $\langle -\lambda \rangle$  or  $\langle \pm \lambda \rangle$ . Here  $\langle x \rangle$  denotes the subgroup of  $\mathbb{R}^*$  generated by  $x$ . In particular  $\mathfrak{diff}_a^1$  is isomorphic to either  $1$ ,  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}$ , or else  $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ .

However, if  $a$  is a left (or right) point of  $C$  then there is no decreasing homeomorphism of  $(\mathbb{R}, C)$  fixing  $a$ . Thus  $\text{Diff}_a^1 = \mathfrak{diff}_a^{1,+}$ , and the result of Lemma 27 holds more generally for  $\text{Diff}_a^1$ .

### 3.2.2 Proof of Theorem 2

We need to show that the identity is an isolated point of the group  $\mathfrak{diff}^1(C)$ , if  $C$  is  $\sigma$ -sparse. To this purpose consider an element  $\mathfrak{diff}^1(C)$  having a representative in  $\psi \in \text{Diff}^1(\mathbb{R}, C)$  such that

$$1 - \sigma < \psi'(x) < 1 + \sigma, \text{ for any } x \in C. \quad (28)$$

There is no loss of generality in assuming that  $\psi \in \text{Diff}^{1,+}(\mathbb{R}, C)$ , i.e. that  $\psi$  is monotone increasing. The minimal element  $\min C$  of  $C$  should therefore be fixed by any element of  $\text{Diff}^{1,+}(\mathbb{R}, C)$ , in particular by  $\psi$ . By Lemma 6,  $\psi \in \text{Diff}_{\min C}^1(\mathbb{R}, C)$  must be 1-flat at  $\min C$ .

Consider the set

$$U = \{x \in C; \psi(z) = z, \text{ for any } z \in C \cap (-\infty, x]\}. \quad (29)$$

The set  $U$  is nonempty, as  $\min C \in U$ . Let  $\xi = \sup U$ .

Assume first that  $\xi$  is not a right point of  $C$ . Since  $\psi$  is continuous,  $\xi \in U$  so that  $\psi \in \text{Diff}_\xi^1$ . From Lemma 6  $\psi'(\xi) = 1$  and  $\psi$  is 1-flat at  $\xi$ . According to Lemma 5 there is some  $\delta > 0$  such that the restriction  $\psi|_{C \cap \mathcal{N}_\delta^+(\xi)}$  is identity, which contradicts the maximality of  $\xi$ .

If  $\xi$  is a right point of  $C$ , then there is some maximal complementary interval  $(\xi, \eta) \subset \mathbb{R} \setminus C$ . Since  $\psi|_{C \cap [\min C, \xi]}$  is identity it follows that  $\psi(C \cap [\xi, \infty)) \subset C \cap [\xi, \infty)$ . As  $\eta$  is the minimal element of  $C \cap (\xi, \infty)$  it should be a fixed point of  $\psi|_{[\xi, \infty)}$  and so  $\eta \in U$ . This contradicts the maximality of  $\xi$ . Hence  $\psi$  is identity on  $C$ .

**Remark 5.** The same arguments show that if  $C \subset [0, 1]$  is a sparse Cantor set and  $\mathfrak{diff}_0^{1,+}(C) = 1$ , then  $\mathfrak{diff}^{1,+}(C) = 1$ .

For the second claim of the theorem let  $V_\delta$  be the set of those elements in  $\mathfrak{diff}_{S^1}^1(C)$  having a representative  $\psi \in \text{Diff}^1(S^1, C)$  such that

$$1 - \delta < \psi'(x) < 1 + \delta, \text{ for any } x \in C. \quad (30)$$

Here elements of  $\text{Diff}^1(S^1)$  are identified with real periodic functions on  $\mathbb{R}$ . We choose  $\delta < \min(\sigma, 0.3)$ . It is enough to prove that  $V_\delta$  is finite.

Consider a complementary interval  $J \subset S^1 - C$  of maximal possible length, say  $|J|$ . Consider its right end  $\eta$ , with respect to the cyclic orientation. If  $\psi \in V_\delta$  is such that  $\psi(\eta) = \eta$ , then the arguments from the proof of Theorem 2 show that  $\psi(x) = x$  when  $x \in C$ .

We claim that the set of intervals of the form  $\psi(J)$ , for  $\psi \in V_\delta$  is finite. Each  $\psi(J)$  is a maximal complementary interval, because if it were contained in a larger interval  $J'$ , then  $\psi^{-1}(J)$  would be a complementary interval strictly larger than  $J$ . This shows that any two such intervals  $\psi(J)$  and  $\varphi(J)$  are either disjoint or they coincide, for otherwise their union would contradict their maximality. Further, each  $\psi(J)$  has length at least  $(1 - \delta)|J|$ . This shows that the set of intervals is a finite set  $\{J_1, J_2, \dots, J_k\}$ .

Assume that  $\psi(J) = \varphi(J)$  and both  $\psi$  and  $\varphi$  preserve the orientation of the circle. If the right end of  $J$  is  $\eta$ , with respect to the cyclic orientation, then  $\varphi \circ \psi^{-1}$  sends  $J$  to  $J$  and hence fixes  $\eta$ . Then the arguments from the proof of Theorem 2 show that  $\varphi \circ \psi^{-1}(x) = x$  when  $x \in C$ . It follows that there are at most  $2k$  elements in  $V_\delta$ , finishing the proof of the first part.

### 3.2.3 Proof of Theorem 3

Let  $C$  be a Cantor set contained within a  $\mathcal{C}^1$ -embedded simple closed curve  $L$  on the orientable manifold  $M$ . For the sake of simplicity we will suppose from now on that  $M$  is a surface, but the proof goes on without essential modifications in higher dimensions. Let  $\varphi$  be a diffeomorphism of  $M$  sending  $C$  into  $C$ . Fix a parameterization of a collar  $N$  such that  $(N, L)$  is identified with  $(L \times [-1, 1], L \times \{0\})$ . Denote by  $\pi : N \rightarrow L$  the projection on the first factor and by  $h : N \rightarrow [-1, 1]$  the projection on the second factor.

There exists an open neighborhood  $U$  of  $C$  in  $L$  so that  $\varphi(U) \subset N$ . In particular, the closure  $\overline{U}$  is a finite union of closed intervals. The map  $\varphi : \overline{U} \rightarrow N = L \times [-1, 1]$  has the property  $h \circ \varphi(a) = 0$ , for each  $a \in C$ . Therefore the differential  $D_a(h \circ \varphi) = 0$ , for each  $a \in C$ . Since  $\varphi$  is a diffeomorphism  $D_a(\pi \circ \varphi) \neq 0$ , for every  $a \in C$ .

For each  $a \in C$  consider an open interval neighborhood  $U_a$  within  $L$ , so that  $D_x(\pi \circ \varphi) \neq 0$  and  $\|D_x(h \circ \varphi)\| < 1$ , for every  $x \in \overline{U_a}$ . We obtain an open covering  $\{U_a; a \in C\}$  of  $C$ . As  $C$  is compact there exists a finite subcovering by intervals  $\{U_1, U_2, \dots, U_n\}$ . Without loss of generality one can suppose that  $U_j \subset U$ , for all  $j$ . We consider such a covering having the minimal number of elements. This implies that  $\overline{U_j}$  are disjoint intervals.

For every  $j$  the map  $\pi|_{\varphi(\overline{U_j})} : \varphi(\overline{U_j}) \rightarrow \pi(\varphi(\overline{U_j})) \subset L$  is a diffeomorphism on its image, since  $\varphi(\overline{U_j})$  is connected and  $D_x(\pi \circ \varphi) \neq 0$ , for any  $x \in \overline{U_j}$ .

Consider a slightly smaller closed interval  $I_j \subset U_j$  such that  $I_j \cap C = U_j \cap C$ .

Let  $\mu$  be a positive smooth function on  $\sqcup_{j=1}^n \overline{U_j}$  such  $\mu(t)$  equals 1 near the boundary points and vanishes on  $\sqcup_{j=1}^n I_j$ . Define  $\phi_s : \sqcup_{j=1}^n \overline{U_j} \rightarrow N$  by:

$$\phi_s(x) = (\pi \circ \varphi(x), (s\mu(x) + 1 - s) \cdot h \circ \varphi(x)). \quad (31)$$

Then  $\phi_0(x) = \varphi(x)$  and for each  $s \in [0, 1]$  we have  $\phi_s(x) = \varphi(x)$ , for  $x$  near the boundary points of  $\sqcup_{j=1}^n \overline{U_j}$ . Furthermore  $\phi_1(x) = \pi \circ \varphi(x) \in L$ , when  $x \in \sqcup_{j=1}^n I_j$ . One should also notice that  $\phi_s(x) = \varphi(x)$ , for each  $x \in C$  and  $s \in [0, 1]$ .

Let now denote  $J_j = \pi \circ \varphi(I_j)$ . It is clear that  $C = \varphi(C) \subset \sqcup_{j=1}^n J_j$ . We claim that we can assume that  $J_j$  are disjoint. Indeed, since  $\varphi$  is bijective we have  $\varphi(I_j \cap C) \cap \varphi(I_k \cap C) = \emptyset$ , for any  $j \neq k$ . Since  $\varphi(I_j \cap C)$  are closed subsets of  $L$  there exists  $\varepsilon > 0$  so that  $d(\varphi(I_j \cap C), \varphi(I_k \cap C)) \geq \varepsilon$ , for  $j \neq k$ , where  $d$  is a metric on  $L$ . Since  $\phi_1(I_j \cap C) = \varphi(I_j \cap C)$ , we have  $d(\phi_1(I_j \cap C), \phi_1(I_k \cap C)) \geq \varepsilon$ , for  $j \neq k$ . Thus there exist some open neighborhoods  $J'_j$  of  $\phi_1(I_j \cap C)$  within  $L$  so that  $d(J'_j, J'_k) \geq \frac{1}{2}\varepsilon$ , for all  $j \neq k$ . As  $\phi_1$  is a diffeomorphism there exist open neighborhoods  $I'_j$  of  $I_j \cap C$  with the property that  $\phi_1(I'_j) \subset J'_j$ , for all  $j$ . Now  $I'_j$  and  $J'_j$  are finite unions of open intervals. We can replace them by closed intervals with the same intersection with  $C$ . This produces two new families of disjoint closed intervals related by  $\phi_1$ , as the initial situation. This proves the claim.

We obtained that there exist two coverings  $\{I_1, I_2, \dots, I_n\}$  and  $\{J_1, J_2, \dots, J_n\}$  of  $C$  by disjoint closed intervals and a diffeomorphism  $\phi_1 : \sqcup_{j=1}^n I_j \rightarrow \sqcup_{j=1}^n J_j$  such that  $\phi_1(x) = \varphi(x)$ , for any  $x \in C$ .

Notice that the sign of  $D_a(\pi \circ \varphi)$  might not be the same for all intervals.

Every partition of  $C$  induced by a covering  $\{I_1, I_2, \dots, I_n\}$  as above is determined by the choice of complementary intervals, namely the  $n - 1$  connected components of  $L \setminus \sqcup_{j=1}^n I_j$ . It follows that there are only countably many finite partitions of  $C$  of the type considered here. Next, the set of those elements of  $\mathfrak{diff}_M^1(C)$  which arise from partitions induced by the coverings  $\{I_1, I_2, \dots, I_n\}$  and  $\{J_1, J_2, \dots, J_n\}$  of  $C$  is acted upon transitively by the stabilizer of one partition. The stabilizer of one partition embeds into the product of  $\mathfrak{diff}_{I_j}^1(C \cap I_j)$ . Theorem 2 then implies that  $\mathfrak{diff}_M^1(C)$  is countable.

### 3.2.4 Proof of Theorem 4

Before to proceed we need some preparatory material. Let  $A \subset \mathbb{R}^n$  be a set without isolated points. Let  $T_p \mathbb{R}^n$  denote the tangent space at  $p$  on  $\mathbb{R}^n$  and  $UT_p \mathbb{R}^n \subset T_p \mathbb{R}^n$  the sphere of unit vectors. For any  $p \in A$  one defines the *unit tangent spread*  $UT_p A \subset UT_p \mathbb{R}^n$  at  $p$  as the set of vectors  $v \in UT_p \mathbb{R}^n$  for which there

exists a sequence of points  $a_i \in A$  with  $\lim_{i \rightarrow \infty} a_i = p$  and

$$\lim_{i \rightarrow \infty} \frac{a_i - p}{\|a_i - p\|} = v.$$

Vectors in  $UT_p A$  will also be called *(unit) tangent vectors* at  $p$  to  $A$ . We also set  $T_p A = \mathbb{R}_+ \cdot UT_p A \subset T_p \mathbb{R}^n$  for the *tangent spread* at  $p$ .

A differentiable map  $\varphi : (\mathbb{R}^n, A) \rightarrow (\mathbb{R}^n, B)$  induces a tangent map  $T_p \varphi : T_p A \rightarrow T_{\varphi(p)} B$ . Specifically, let  $D_p \varphi : T_p \mathbb{R}^n \rightarrow T_{\varphi(p)} \mathbb{R}^n$  be the differential of  $\varphi$ ; then we have

$$T_p \varphi = U(D_p \varphi),$$

where for a linear map  $L : V \rightarrow W$  between vector spaces we denoted by  $U(L) : U(V) \rightarrow U(W)$  the map induced on the unit spheres, namely

$$U(L)v = \frac{L(v)}{\|L(v)\|}.$$

As the unit tangent spread  $UT_p A$  is a subset of the unit sphere, it inherits the spherical geometry and metric. In particular, it makes sense to consider the convex hull  $Hull(UT_p A) \subset UT_p \mathbb{R}^n$  in the sphere.

Although tangent spreads to product Cantor sets might depend on the particular factors, their convex hulls have a simple description. Let  $C = C_1 \times C_2 \times \cdots \times C_n \subset \mathbb{R}^n$  be a product of Cantor sets  $C_i \subset \mathbb{R}$ . The usual cubical complex underlying the  $n$ -dimensional cube  $[0, 1]^n$  will be denoted by  $\square^n$ . Let then denote by  $Lk(p)$  the spherical link of  $p \in \square^n$ . If  $p$  belongs to a  $k$ -dimensional cube but not to a  $k+1$ -dimensional cube of  $\square^n$  then  $Lk(p)$  is isometric to the link  $\mathcal{L}_{k,n}$  of the origin in  $\mathbb{R}^k \times \mathbb{R}_+^{n-k}$ . Thus there are precisely  $n+1$  different isometry types of links of points.

Now a direct inspection shows that for each  $p \in C$  there exists some  $k$  so that the convex hull  $Hull(UT_p A)$  is isometric to  $\mathcal{L}_{k,n}$ .

When the diffeomorphism  $\varphi : (\mathbb{R}^n, C) \rightarrow (\mathbb{R}^n, C)$  is also *conformal*, then the tangent maps are isometries between the unit tangent spreads, because the spherical distance is given by angles between the corresponding vectors. However this is not true for general diffeomorphisms.

Nevertheless the spherical links  $\mathcal{L}_{k,n}$  are quite particular. There exist  $n+k$  vectors along the coordinates axes which are extremal points of  $UT_p C$ , such that their convex hull is  $Hull(UT_p C)$ , so isometric to  $\mathcal{L}_{k,n}$ . These are vectors of the form  $e_i, -e_i, e_j$ , where  $e_i$  correspond to the coordinates axes in  $\mathbb{R}^k$  and  $e_j$  to those in  $\mathbb{R}^{n-k}$ . Now, any diffeomorphism  $\varphi : (\mathbb{R}^n, C) \rightarrow (\mathbb{R}^n, C)$  should send a unit tangent spread of type  $\mathcal{L}_{k,n}$  into one of the same type, since  $\mathcal{L}_{k,n}$  is not affinely equivalent to  $\mathcal{L}_{k',n}$ , for  $k \neq k'$ . Moreover, the extremal vectors are sent into extremal vectors of the same type.

Let further  $\varphi \in \text{Diff}^1(\mathbb{R}^n, C)$  such that  $\|D_a \varphi - \mathbf{1}\| \leq \varepsilon$  for all  $a \in C$ . Assume now that the unit tangent spread  $UT_a C$  is isometric to  $\mathcal{L}_{0,n}$ , namely it is of *corner* type. In this case  $U(D_a \varphi)$  should permute the  $n$  coordinate vectors, which are the extremal vectors of  $\mathcal{L}_{0,n}$ . Therefore either  $U(D_a \varphi) = \mathbf{1}$ , or else

$$\|U(D_a \varphi) - \mathbf{1}\| \geq \sqrt{2},$$

which yields

$$\|D_a \varphi - \mathbf{1}\| \geq \sqrt{2}.$$

In other words, taking  $\varepsilon < \sqrt{2}$  any diffeomorphism  $\varphi$  as above should satisfy  $U(D_a \varphi) = \mathbf{1}$ . Now, if  $\varphi$  is of class  $\mathcal{C}^1$  then  $U(D_a \varphi)$  is continuous. Since the set of corner points is dense in  $C$  we derive  $U(D_a \varphi) = \mathbf{1}$ , for any  $a \in C$ . This is the same as saying that for any  $a \in C$  the linear map  $D_a \varphi$  is represented by a diagonal matrix, with respect to the standard coordinate system of  $\mathbb{R}^n$ .

**Proposition 1.** *Let  $a \in C$  be a corner point. The map  $\chi : \text{Diff}_{\mathbb{R}^n, a}^1(C) \rightarrow (\mathbb{R}^*)^n$ , which associates to the germ  $\varphi$  the eigenvalues of  $D_a \varphi$  is an isomorphism onto a discrete subgroup of  $(\mathbb{R}^*)^n$ .*

*Proof.* Let  $\{x_1, x_2, \dots, x_n\}$  be the standard coordinates functions on  $\mathbb{R}^n$  and  $\pi_j : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  denote the projection onto the hyperplane  $H_j = \{x_j = 0\}$ . For the sake of simplicity we assume that  $a = (0, 0, \dots, 0)$ , and that the convex hull of the unit tangent spread is the union of the sets  $H_j^+ = H_j \cap \{x_i \geq 0, i = 1, \dots, n\}$ . We will use induction on  $n$ . The claim was proved in Lemma 7 for  $n = 1$ . Assume it holds for all dimensions at most  $n - 1$ .

Let  $\varphi \in \text{Diff}^1(\mathbb{R}^n, C)$  such that  $\varphi(a) = a$ . Assume that  $\|D_x\varphi - \mathbf{1}\| < \frac{1}{2}\sigma < \frac{1}{2}$  for all  $x$  in a neighborhood  $V$  of  $a$  in  $\mathbb{R}^n$ . We will prove that  $\varphi|_C$  is a trivial germ at  $a$ . This shows that the image of  $\chi$  is a discrete subgroup of  $(\mathbb{R}^*)^n$  and the kernel of  $\chi$  is trivial.

Consider the maps  $\varphi_j : H_j \rightarrow H_j$  given by  $\varphi_j(x) = \pi_j \circ \varphi(x)$ . The determinant of  $D_a\varphi_j$  is the product of all eigenvalues of  $D_a\varphi$  but the  $j$ -th eigenvalue, and hence it is non-zero. Moreover, we have  $\|D_a\varphi_j - \mathbf{1}\| < \frac{1}{2}\sigma$ .

We claim that

**Lemma 8.** *The map  $\varphi_j : H_j \cap V \rightarrow H_j$  is injective.*

*Proof.* Assume the contrary, namely that there exist two points  $p, q \in H_j \cap V$  such that  $\pi_j(\varphi(p)) = \pi_j(\varphi(q))$ . Consider the first non-trivial case  $n = 2$ , when  $H_j^+$  are half-lines issued from  $a$ . The mean value theorem and the previous equality prove that there exists some  $\xi \in H_j^+ \cap V$  between  $p$  and  $q$  so that  $(\pi_j \circ \varphi)'(\xi) = 0$ . This amounts to the fact that the image of  $D_\xi\varphi$  is contained in the kernel of  $D_{\varphi(\xi)}\pi_j$ , namely that

$$\langle D_\xi\varphi(v_j), v_j \rangle = 0,$$

where  $v_j$  is a unit tangent vector to  $H_j^+$  at  $\xi$ . We derive  $\|D_\xi\varphi_j - \mathbf{1}\| \geq 1$ , contradicting our assumptions.

In the general case  $n > 2$  we will use a trick to reduce ourselves to  $n = 2$ , because we lack a multidimensional mean value theorem. Let  $P$  a generic affine 2-dimensional half-plane whose boundary line passes through  $p$  and  $q$ . We can find arbitrarily small  $\mathcal{C}^1$ -isotopy deformations  $\psi$  of  $\varphi_j$  so that  $\psi(H_j)$  is transversal to  $P$  and  $\|D_a\psi - \mathbf{1}\| < \sigma$ . It follows that  $\psi(H_j) \cap P$  is a 1-dimensional manifold  $Z$  with boundary containing both  $p$  and  $q$ . Now either there exist two distinct points of the boundary  $\partial Z$  joined by an arc within  $Z$ , or else there is an arc of  $Z$  issued from  $p$  which returns to  $p$ , contradicting the transversality of the intersection  $\psi(H_j) \cap P$ . In any case the mean value argument above shows that it should exist a point  $\psi(\xi)$  of  $Z$  for which the tangent vector  $v$  is orthogonal to  $H_j$ . We can write  $v = D_\xi\psi(w)$ , for some tangent vector  $w \in H_j$  at  $\xi$ . It follows that

$$\langle D_\xi\psi(w), w \rangle = 0,$$

which implies  $\|D_\xi\psi - \mathbf{1}\| \geq 1$ , contradicting our assumptions. □

It follows that  $\varphi_j : H_j \cap V \rightarrow H_j$  is an injective map of maximal rank in a neighborhood  $V$  of  $a$ , and hence a diffeomorphism on its image. The projection  $\pi_j$  sends  $C$  into  $C \cap H_j$ , so that

$$\varphi_j(C \cap H_j \cap V) \subset C \cap \varphi(H_j \cap V) \subset C \cap H_j.$$

Our aim is to use the induction hypothesis for  $\varphi_j$ . In order to do that we need to show that the class of  $\varphi_j$  defines indeed an element of  $\text{Diff}_{\mathbb{R}^{n-1}, a}^1(C)$ , where we identified  $H_j$  with  $\mathbb{R}^{n-1}$ .

We assume from now on that the neighborhood  $V$  is a parallelepiped, all whose vertices being corner points. Its boundary  $\partial V$  will then consist of the union of the faces  $V_j = \partial V \cap H_j^+$  with their respective parallel faces  $V_j'$ . The parallelepiped  $V$  is surrounded by gaps, whose smaller width is some  $\delta > 0$ . Let  $V^\delta$  be the  $\delta$ -neighborhood of  $V$ . If  $\varphi$  is Lipschitz with Lipschitz constant  $1 + \varepsilon$  and

$$(1 + \varepsilon)l_i < \delta + l_i$$

where  $l_i$  are the edges lengths of  $V$  then the image  $\varphi(V)$  is contained in  $V^\delta$ , so that  $\varphi_j(V_j) \subset V^\delta \cap H_j$ .

Further  $\varphi_j(\partial V_j)$  bounds  $\varphi_j(V_j)$  and thus there are no points of  $C \cap H_j$  accumulating on  $\varphi_j(V_j)$ , as their unit tangent spread cannot be of the type  $\mathcal{L}_{n-1, n-1}$ . Thus  $C - \varphi_j(V_j)$  is a closed subset of  $C$  and hence its distance to  $\varphi_j(V_j)$  is strictly positive. There exists then an open set  $U \subset V^\delta$  which contains  $\varphi_j(V_j)$  such that  $U \cap (C - \varphi_j(V_j)) = \emptyset$ . It follows that there exists an extension of  $\varphi_j$  to a diffeomorphism  $\Phi_j$  of  $(H_j, C)$  which is identity outside  $U$ , and hence on  $(V_\delta \cap H_j) \cup (C - \varphi(V_j))$ .

It only remains to check that  $\Phi_j^{-1}(C)$  is also contained in  $C$ , as needed for  $\Phi_j \in \text{Diff}^1(\mathbb{R}^{n-1}, C)$ . This follows from the following:

**Lemma 9.** *The map  $\varphi_j$  has the property*

$$\varphi_j(C \cap V_j) = C \cap \varphi(V_j).$$

*Proof.* Assume that there exists some point  $p$  in  $\varphi(V_j) \cap C$  which does not belong to  $V_j$ . Then the line issued from  $p$  which is orthogonal to  $V_j$  intersects  $\varphi(V_j)$  only once, from Lemma 8. On the other hand there are points of  $C$  on this line, as  $C$  is a product and  $p \notin V_j$ . By Jordan's theorem there exist points of  $C$  which belong to different components of  $R^n - \varphi(\partial V)$  which contradicts the fact that  $\varphi$  is surjective on  $C$ .

Thus  $\varphi(C \cap V_j) \subset C \cap V_j$ . The same argument for  $\varphi^{-1}$  yields the opposite inclusion and hence  $\varphi(C \cap V_j) = C \cap V_j$ . Our claim follows.  $\square$

Lemma 9 tells us that  $\varphi_j$  defines a germ in  $\mathfrak{diff}_{H_j, a}^1(C \cap H_j)$ , namely both  $\varphi_j$  and  $\varphi_j^{-1}$  sends  $C \cap H_j$  into itself. By the induction hypothesis  $\varphi_j|_{C \cap H_j}$  must be identity in a neighborhood of  $a$  within  $H_j$ .

Notice that this implies already that  $D_a \varphi = \mathbf{1}$ , and hence establishing the first claim of Proposition 1.

For the second claim we consider the distance  $d(C - V, V) = \mu > 0$ , as  $V$  is surrounded by gaps. We suppose further that

$$\|D_x \varphi - \mathbf{1}\| < \min(\sigma/2, \frac{\mu}{1+\sigma}).$$

We know that  $\varphi(y, 0) = (y, u(y))$ , for  $y \in C \cap V \cap H_n$  and some function  $u \geq 0$ . The next step is to show that  $u|_{C \cap V \cap H_n} = 0$ .

Assume that there exists some  $x \in C \cap V \cap H_n$  so that  $u(x) > 0$ . Observe that  $u(x) \in C_n$ , since  $\varphi(C) \subset C$ . Since points of  $C_n$  which are not endpoints are dense in  $C_n$  there should exist  $x \in C$  for which  $u(x)$  is not an endpoint of  $C_n$ . Set  $z = (x, u(x)) \in C$ .

Then for each  $\nu > 0$  there exist points  $z_+, z_- \in C$  with  $\pi_n(z_+) = \pi_n(z_-) = x$ , so that the distances  $d(z_+, z), d(z_-, z) < \nu$ .

Observe that the segment  $z_+ z_-$  intersects just once  $\varphi(H_n^+)$ , namely at  $z$ . One might expect to use Jordan's theorem in order to derive that  $z_+ \in C$  and  $z_- \in C$  could not belong to the same connected component of  $\varphi(\partial V)$ . This is not exactly true, as the segment  $z_+ z_-$  could possibly intersect other sheets like  $\varphi(H_j^+)$  which are part of  $\varphi(\partial V)$ .

Set  $r$  for the distance between  $x \in H_n^+$  and the union of the other  $2n - 1$  faces of  $\partial V$ . By the induction hypothesis we can assume that  $r > 0$ . Choose now  $\nu$  so that  $\nu < \min((1 - \sigma)r/2, \mu(1 - \sigma)/2)$ .

Suppose that there exist  $x_+, x_- \in C \cap V$  such that  $\varphi(x_+) = z_+$  and  $\varphi(x_-) = z_-$ . By Jordan's theorem the segment  $z_+ z_-$  intersects at least once  $\varphi(\partial V - H_n^+)$ , say in a point  $\tilde{z} = \varphi(\tilde{x})$ .

We have then  $d(x, \tilde{x}) \geq r$ , while

$$d(\varphi(x), \varphi(\tilde{x})) \leq d(z_+, z_-) \leq 2\nu.$$

On the other hand the  $\mathcal{C}^1$ -diffeomorphism  $\varphi^{-1}$  is Lipschitz with Lipschitz constant bounded by  $\sup_{x \in V} \|D_x \varphi^{-1}\|$ . Now, by standard functional calculus we have:

$$\|D_x \varphi^{-1}\| \leq \sum_{k=0}^{\infty} \|\mathbf{1} - D_x \varphi\|^k < \frac{1}{1 - \sigma}.$$

Therefore the Lipschitz constant of  $\varphi^{-1}$  is bounded by above by  $\frac{1}{1 - \sigma}$  so that

$$d(x, \tilde{x}) \leq \frac{1}{1 - \sigma} d(\varphi(x), \varphi(\tilde{x})) \leq \frac{2\nu}{1 - \sigma}.$$

This contradicts our choice of  $\nu$ .

Furthermore if one of  $x_+, x_-$ , say  $x_+$  belongs to  $C - V$  then we have  $d(x, x_+) \geq \mu$  while

$$d(\varphi(x), \varphi(x_+)) \leq \nu$$

and the argument above still leads to a contradiction.

This shows that  $\varphi$  cannot be surjective on  $C$ . On the other hand a diffeomorphism of  $\mathbb{R}^n$  which preserves  $C$  restricts to a bijection on  $C$ . If it were not surjective then its inverse would send points of  $C$  outside.



In particular  $u(x)|_{C \cap H_n^+} = 0$  and so  $\varphi|_{C \cap H_n^+}$  is identity. The same proof shows that  $\varphi|_{C \cap H_j^+}$  is identity, for all  $j$ .

By using the same argument when  $a$  runs over the points of  $V \cap C \cap \bigcup_{j=1}^n H_j^+$  we derive that  $\varphi|_{C \cap V}$  is identity, as claimed.  $\square$

*End of the proof of theorem 4.* The proof is by induction on  $n$ . For  $n = 1$  this was already proved above. Let  $V$  denote now the smallest parallelepiped containing  $C$ , in order to match previous notations and constructions. Suppose that  $\varphi \in \text{Diff}^1(\mathbb{R}^n, C)$  is such that  $\|D_x \varphi - \mathbf{1}\| < \varepsilon$ , for all  $x \in V^\delta$ . Then  $\varphi(\partial V)$  surrounds  $C$  and the proof of Lemma 9 gives us  $\varphi(C \cap \partial V) = C \cap \partial V$ . Moreover, each  $\varphi_j$  preserves the associated face  $V_j$ . By the induction hypothesis  $\varphi_j$  is identity. It follows that  $\varphi|_{C \cap \partial V}$  is identity. We can therefore use Proposition 1 to derive that around every corner point of  $C \cap \partial V$  the map  $\varphi|_C$  is identity. The same argument works for all corner points of  $V$ .

**Remark 6.** If  $C = C_\lambda^n$ , then  $\text{diff}_a^{1,+}(C_\lambda)$  is isomorphic to  $\mathbb{Z}^{r(a)}$ , where  $r(a)$  is the number of coordinates of  $a$  which are  $\lambda$ -rational (compare with [4]).

## 4 Diffeomorphisms groups of specific Cantor sets

### 4.1 Proof of Theorem 5

Observe first that  $C_\Phi$  is a Cantor set. Indeed the contractivity assumption implies that an infinite intersection  $\lim_{p \rightarrow \infty} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_p}(M)$  cannot contain but a single point. Two such points which are distinct are separated by some smoothly embedded sphere, which is the image of  $\partial M$  by an element of the semigroup generated by  $\Phi$ , so that the set  $C_\Phi$  is totally disconnected. The perfectness follows the same way.

We will draw a rooted  $(n+2)$ -valent tree  $\mathcal{T}$  with edges directed downwards. When  $M = [0, 1]$  there is an extra structure on  $\mathcal{T}$ , as all edges issued from a vertex are enumerated from left to the right.

There is a one-to-one correspondence between the points of the boundary at infinity of the tree and the points of the Cantor set  $C = C_\Phi$  associated to the invertible IFS  $(\Phi, M)$ . To each point  $\xi \in C$  we can assign an infinite sequence  $I = i_1 i_2 \dots i_p \dots$ , so that  $\xi = \xi(I)$  where we denoted:

$$\xi(I) = \bigcap_{p=1}^{\infty} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_p}(M).$$

The vertices of the tree are endowed with a compatible labeling by means of finite multi-indices  $I$ , where the root has associated the empty index and the vertex  $v_I$  is the one reached after traveling along the edges labeled  $i_1, i_2, \dots, i_p$ . We also put

$$\phi_I(x) = \phi_{i_1} \phi_{i_2} \cdots \phi_{i_p}(x)$$

for finite  $I$ . This extends obviously to the case of infinite multi-indices  $I$ .

We further need to introduce a special class of germs, as follows:

**Definition 9.** The standard germ associated to the finite multi-indices  $I$  and  $J$  is the diffeomorphism  $\phi_{I/J} : \phi_I(M) \rightarrow \phi_J(M)$  given by

$$\phi_{I/J}(\phi_I(x)) = \phi_J(x). \quad (32)$$

Standard germs preserve the Cantor set  $C$  as germs, namely  $\phi_{I/J}(C \cap \phi_I(M)) \subset C \cap \phi_J(M)$ . In fact if  $S$  is an infinite multi-index then

$$\phi_{I/J}(\xi(IS)) = \xi(JS).$$

Graphically we can realize this map as a partial isomorphism of the tree  $\mathcal{T}$  which maps the subtree hanging at  $v_I$  onto the subtree hanging at  $v_J$ .

Consider a pair  $(t_1, t_2)$  of finite labeled subtrees of the same degree of  $\mathcal{T}$  both containing the root, and whose leaves are enumerated  $v_{I_1}, v_{I_2}, \dots, v_{I_p}$  and  $v_{J_1}, v_{J_2}, \dots, v_{J_p}$ .

**Lemma 10.** Assume that  $\phi_j$  are orientation preserving diffeomorphisms of  $M$ . Then the map

$$\phi(x) = \phi_{I_k/J_k}(x), \quad \text{if } x \in \phi_{I_k}(C) \quad (33)$$

defines an element  $\phi_{(t_1, t_2)} \in \mathfrak{diff}^{1,+}(C)$ .

*Proof.* We know that  $C = \cup_{i=0}^n \phi_i(C)$ , since  $C$  is the attractor of  $\Phi$ . By recurrence on the number of leaves we show that

$$C = \cup_{i=0}^n \phi_{I_i}(C)$$

for any finite subtree  $t$  of  $\mathcal{T}$  containing the root and having leaves  $v_{I_i}$ ,  $i = 0, n$ . Now  $\phi$  is a smooth orientation preserving map defined on  $\cup_{i=0}^n \phi_{I_i}(M)$ , and so its domain of definition contains  $C$ .

When the dimension  $d = 1$ , the complementary  $M \setminus \cup_{i=0}^n \phi_{I_i}(M)$  is the union of finitely many intervals, which we call gaps and there exists by orientability assumption an extension of  $\phi$  to a diffeomorphism of  $M = [0, 1]$  sending gaps into gaps.

When the dimension  $d > 1$ , the complementary gap  $M \setminus \cup_{i=0}^n \phi_{I_i}(M)$  is now connected and diffeomorphic to the standard disk with  $(n + 1)$  holes. Moreover, the restriction of  $\phi$  to every sphere  $\partial\phi_{I_i}(M)$  is isotopic to identity since it is orientation preserving and it admits an extension to the ball. Taking a suitable smoothing at the singular vertex of the conical extension of  $\phi|_{\cup_{i=0}^n \phi_{I_i}(\partial M)}$  we obtain an extension of  $\phi$  to a diffeomorphism of the ball  $M$ , possibly non-trivial on  $\partial M$ .

This extension preserves  $C$  invariant as gaps are disjoint from  $C$  and therefore defines an element  $\phi_{(t_1, t_2)} \in \mathfrak{diff}^{1,+}(C)$ .  $\square$

*End of the proof of Theorem 5.* Let us stabilize the pair of trees  $(t_1, t_2)$  to a pair  $(t'_1, t'_2)$ , where  $t'_j$  is obtained from  $t_j$  by adding the first descendants at vertex  $v_{I_s}$ , for  $j = 1$  and  $v_{J_s}$ , when  $j = 2$ . The new vertices come with a compatible labeling. Moreover, an orientation preserving diffeomorphism of  $C$  induces a monotone map of the boundary of the tree, when  $d = 1$ .

By direct inspection using the explicit form of  $\phi$  we find that:

$$\phi_{(t_1, t_2)} = \phi_{(t'_1, t'_2)}.$$

Thus the map which associates to the pair  $(t_1, t_2)$  of labeled trees the element  $\phi_{(t_1, t_2)}$  factors through a map  $F_{n+1} \rightarrow \mathfrak{diff}^{1,+}(C)$ , for  $d = 1$ , and  $V_{n+1} \rightarrow \mathfrak{diff}^{1,+}(C)$ , for  $d = 2$ , respectively. This is easily seen to be a homomorphism. When  $I \neq J$  the map  $\varphi_{I/J}|_C$  is not identity since  $\varphi_I(M) \cap \varphi_J(M) = \emptyset$ . This proves that the homomorphism defined above is injective, thereby ending the proof of Theorem 5.

**Remark 7.** There is a more general setting in which we allow basins to have boundary fixed points. We say that the compact submanifold  $M$  is an attractive basin for  $\Phi = (\phi_0, \phi_1, \dots, \phi_n)$  if, for all  $j \in \{0, 1, \dots, n\}$  we have:

1.  $\phi_j(\text{int}(M)) \subset \text{int}(M)$ ;
2.  $\text{int}(\phi_j^{-1}(\phi_j(\partial M) \cap \partial M)) \supset \text{int}(\phi_j(\partial M) \cap \partial M)$ ;
3.  $\phi_i(M) \cap \phi_j(M) = \emptyset$ , for any  $i \neq j \in \{0, 1, \dots, n\}$ ;
4.  $\text{int}(\phi_j(\partial M) \cap \partial M) \subset \text{int}(\phi_j^{-1}(\phi_j(\partial M) \cap \partial M))$ .

Using similar arguments one can show that  $\mathfrak{diff}^1(C_\Phi)$  contains  $F_{n+1}$  whenever  $\Phi$  has an attractive basin.

**Remark 8.** If the Cantor set  $C$  is invertible, namely there exists an orientation reversing diffeomorphism  $\phi$  of  $M$  preserving  $C$ , then we can replace the homeomorphisms  $\phi_j$  which reverse the orientation by  $\phi \circ \phi_j$ . However, there exist non invertible Cantor subsets, for instance the union of two copies  $C_\lambda \cup (1 + C_\mu)$ , for  $\lambda \neq \mu$ .

## 4.2 Proof of Theorem 6 for $C = C_\lambda$

Our strategy is to give first a detailed proof of Theorem 6 in the case when  $C = C_\lambda$  and then to explain the necessary changes needed to achieve the general case in the next section.

We first need the following:

**Lemma 11.** *If  $a$  is a left (or right) point of  $C_\lambda$ , then  $\chi(\text{Diff}_a^1)$  is the subgroup  $\langle \lambda \rangle \subset \mathbb{R}^*$ .*

*Proof.* Recall from Remark 4 that  $\text{Diff}_a^1(C_\lambda) = \text{Diff}_a^{1,+}(C_\lambda)$ . The set  $L(C_\lambda)$  of left points of  $C_\lambda$  is affinely locally homogeneous, namely for any two left points  $a$  and  $b$  there exists an affine germ sending a neighborhood of  $a$  in  $C_\lambda$  into a neighborhood of  $b$  in  $C_\lambda$ . Therefore it suffices to analyze  $\text{Diff}_0^{1,+}(C_\lambda)$ . Moreover, 0 is the minimal element of  $C_\lambda$  and therefore it should be fixed by any element of  $\text{Diff}_0^{1,+}(C_\lambda)$ .

Elements of  $L(C_\lambda)$  can be described explicitly, as:

$$L(C_\lambda) = \bigcup_{n=1}^{\infty} \{x \in [0, 1]; x = \sum_{j=1}^n a_j \lambda^{-j}, \text{ where } a_j \in \{0, \lambda - 1\}\}. \quad (34)$$

Therefore there exists  $\delta$  such that the multiplication by  $\lambda \in \mathbb{R}^*$  sends  $C_\lambda \cap \mathcal{N}_\delta(0)$  into  $C_\lambda$ . This easily implies that  $\chi(\text{Diff}_a^{1,+})$  contains the subgroup  $\langle \lambda \rangle$ .

For the reverse inclusion we need a sharpening of Lemma 6. Note first that the set of lengths of gaps in  $C_\lambda$  is  $\{(\lambda - 2)\lambda^{-n}, n \in \mathbb{Z}_+ \setminus \{0\}\}$ . In particular, the quotients of the lengths of any two gaps belong to  $\langle \lambda \rangle$ .

Let now  $\alpha > 1$  minimal which occurs in  $\chi(\text{Diff}_0^{1,+}(C_\lambda)) \subset \mathbb{R}_+^*$ . By Lemma 7 there exists  $k \in \mathbb{Z}_+$  such that  $\lambda^{-1} = \alpha^k$ . Let  $\varphi \in \text{Diff}_0^{1,+}$  such that  $\varphi'(0) = \lambda^{-1/k}$ . Consider a set of maximal gaps  $(x_n, y_n)$  accumulating to 0. This means that  $(x_{n+1}, y_{n+1})$  is a maximal gap within  $[0, x_n]$ , for every  $n$ . Then  $(\varphi(x_n), \varphi(y_n))$  is also a maximal gap in  $[0, \varphi(x_{n-1})]$ , so that

$$\frac{\varphi(x_n) - \varphi(y_n)}{x_n - y_n} \in \langle \lambda \rangle.$$

On the other hand

$$\lim_{n \rightarrow \infty} \frac{\varphi(x_n) - \varphi(y_n)}{x_n - y_n} = \varphi'(0) = \lambda^{-1/k}$$

Thus  $\lambda^{-1/k} \in \langle \lambda \rangle$ , so that  $k = 1$  and hence  $\alpha = \lambda^{-1}$ .  $\square$

We next observe that for each left point  $a$  of  $C_\lambda$  there exists a small neighborhood  $U_a$  of  $a$  such that the affine map  $\psi_a = a + \lambda(x - a)$  sends  $U_a \cap C_\lambda$  into  $C_\lambda$ , defining therefore a germ in  $\mathfrak{diff}_a^{1,+}$ . Then Lemmas 11 and 7 imply together that  $\mathfrak{diff}_a^{1,+}$  is generated by  $\psi_a = a + \lambda(x - a)$ .

Let  $a$  and  $b$  be two left points of  $C_\lambda$ . Denote by  $D(a, b)$  the set of germs at  $a$  of classes of local diffeomorphisms  $\varphi$  of  $(\mathbb{R}, C_\lambda)$  such that  $\varphi(a) = b$ . Then  $D(a, b)$  is acted upon transitively by  $\mathfrak{diff}_a^{1,+}$ . Using an argument similar to the one from above concerning stabilizers,  $D(a, b)$  consists of germs of maps of the form  $\psi_{a,b,k} = b + \lambda^k(x - a)$ , with  $k \in \mathbb{Z}$ .

Let now  $\varphi \in \text{Diff}^{1,+}(\mathbb{R}, C_\lambda)$  such that  $\varphi(a) = b$ . From above there exists  $\delta > 0$  such that  $\varphi|_{C_\lambda \cap \mathcal{N}_\delta(a)}$  coincides with  $\psi_{a,b,k}|_{C_\lambda \cap \mathcal{N}_\delta(a)}$  and hence  $\varphi'(a) \in \langle \lambda \rangle$ . Therefore, for any left point  $a \in C_\lambda$  we have  $\varphi'(a) \in \langle \lambda \rangle$ . Now, left points of  $C_\lambda$  are dense in  $C_\lambda$ ,  $\varphi'$  is continuous and  $\langle \lambda \rangle$  has no other accumulation points in  $\mathbb{R}^*$ . It follows that  $\varphi'(a) \in \langle \lambda \rangle$ , for any  $a \in C_\lambda$  and any  $\varphi \in \text{Diff}^{1,+}(\mathbb{R}, C_\lambda)$ .

For a given  $\varphi \in \text{Diff}^{1,+}(\mathbb{R}, C_\lambda)$  its derivative  $\varphi'$  is continuous on the whole interval  $[0, 1]$  and hence is bounded. Moreover, the same argument for  $\varphi^{-1}$  shows that  $\varphi'$  is also bounded from below away from 0, so that  $\varphi'|_{C_\lambda}$  can only take finitely many values of the form  $\lambda^n$ ,  $n \in \mathbb{Z}$ .

The following is a key ingredient in the description of the group  $\mathfrak{diff}^{1,+}(C_\lambda)$ :

**Lemma 12.** *Let  $\varphi \in \mathfrak{diff}^{1,+}(C_\lambda)$ . There is a covering of  $C_\lambda$  by a finite collection of disjoint closed intervals  $I_k$ , such that  $\varphi|_{C_\lambda \cap I_k}$  is the restriction of an affine function to  $I_k \cap C_\lambda$ . Specifically,*

$$\varphi(x) = \varphi(c_k) + \lambda^{j_k}(x - c_k), \quad \text{for } x \in I_k \cap C_\lambda, \quad (35)$$

where  $c_k$  is a left point of  $C_\lambda \cap I_k$ .

*Proof.* For  $c \in C_\lambda$  there is some  $m \in \mathbb{Z}$  such that  $\varphi'(c) = \lambda^m$ . We want to prove that there exists an open neighborhood  $U$  of  $c$  such that:

$$\varphi(x) = \varphi(c) + \lambda^{jk}(x - c), \quad \text{for } x \in U \cap C_\lambda. \quad (36)$$

Then such neighborhoods will cover  $C_\lambda$  and we can extract a finite subcovering by clopen (closed and open) subsets with the same property.

This claim is true for any left (and by similar arguments for right) end points  $c$  of  $C_\lambda$ . It is then sufficient to prove that whenever we have a sequence of left points  $a_n \rightarrow a_\infty$  contained in a closed interval  $U \subset [0, 1]$  and a  $\mathcal{C}^1$ -diffeomorphism  $\varphi : U \rightarrow \varphi(U) \subset [0, 1]$  with  $\varphi(C \cap U) \subset C$ , there exists a neighborhood  $U_{a_\infty}$  of  $a_\infty$  and an affine function  $\psi$  such that for large enough  $n$  the following holds:

$$\varphi(x) = \psi(x), \text{ for } x \in C_\lambda \cap U_{a_\infty}.$$

Around each left point  $a_n$  there are affine maps  $\psi_{a_n, k_n} : U_{a_n, k_n} \rightarrow [0, 1]$  defining germs in  $D(a_n, c_n)$ , where  $c_n = \varphi(a_n)$ , such that  $c_n$  converge to  $c_\infty = \varphi(a_\infty)$  and

$$\varphi(x) = \psi_{a_n, k_n}(x), \text{ for } x \in C_\lambda \cap U_{a_n, k_n}.$$

We can further assume that  $U_{a_n, k_n} \cap C_\lambda$  are clopen sets and we can take  $U_{a_n, k_n} = [a_n, b_n]$  where  $b_n$  are right points of  $C_\lambda$ , and the sequence  $a_n$  is monotone, say increasing.

There is no loss of generality to assume that  $\psi'_{a_n, k_n}|_{C \cap U_{a_n, k_n}}$  is independent on  $n$ , say it equals  $\lambda^m$ , namely  $k_n = m$ . Replacing  $\varphi$  by its inverse  $\varphi^{-1}$  we can also assume that  $m \leq 0$ . Since  $C_\lambda$  is invariant by the homothety of factor  $\lambda$  and center 0, we can further reduce the problem to the case where  $m = 0$ . We have then  $\varphi'(a_\infty) = 1$ , by continuity.

Choose  $n$  large enough so that  $|\varphi'(x) - 1| < \varepsilon$ , for any  $x \in [a_n, a_\infty]$ , where the exact value of  $\varepsilon$  will be chosen later. Let now consider the maximal interval of the form  $[a_n, b]$  to which we can extend  $\psi_{a_n, 0}$  to an affine function which coincides with  $\varphi$  on  $C \cap [a_n, b]$ .

If  $b = a_\infty$ , then the Lemma follows. Otherwise, it is no loss of generality in assuming that  $b = b_n$  and thus  $b$  is a right point of  $C_\lambda$ . Then  $b_n$  is adjacent to some gap  $(b_n, d)$ . Since  $d$  is a left point of  $C_\lambda$  and  $\varphi'(d) = 1$ , we can suppose that  $d = a_{n+1}$ .

Since  $\varphi$  preserves  $C \cap U$ , it should send the gap  $(b_n, a_{n+1})$  into some gap contained into  $[\varphi(a_n), \varphi(b_{n+1})]$ . Recall from above that the ratios of lengths of gaps of  $C_\lambda$  is the discrete subset  $\langle \lambda \rangle \subset \mathbb{R}^*$ . When  $|\varphi'(x) - 1| < \varepsilon$ , we derive that the ratio of the lengths of the gaps  $\varphi(b_n, a_{n+1})$  and  $(b_n, a_{n+1})$  is bounded by  $1 + \varepsilon$ . By taking  $\varepsilon < 1 - \lambda$  we see that the only possibility is that the lengths of these two gaps coincide, namely that

$$\varphi(a_{n+1}) = \varphi(b_n) + a_{n+1} - b_n.$$

This implies that there is a smooth extension of  $\psi_{a_n, 0}$  to an affine function on  $[a_n, b_{n+1}]$  which coincides with  $\varphi$  on points of  $C_\lambda$ , contradicting the maximality of  $b = b_n$ . This proves that  $b = a_\infty$ , proving the claim.

When  $a_\infty$  is not a right point we also have an affine extension of  $\varphi$  to a right neighborhood of  $a_\infty$ , by the same argument.  $\square$

Consider the rooted binary tree  $\mathcal{T}$  embedded in the plane so that its ends abut on the interval  $[0, 1]$ . We label each edge  $e$  by  $l(e) \in \{0, \lambda - 1\}$ , such that the leftmost edge is always labeled 0. Let  $v$  be a vertex of  $\mathcal{T}$  and  $e_1, e_2, \dots, e_n$  the sequence of edges representing the geodesic which joins the root to  $v$ . To the vertex  $v$  one associates then the number

$$r(v) = \sum_{j=1}^n l(e_j) \lambda^{-j}. \quad (37)$$

Denote by  $D(v)$  the set of all descendants of the vertex  $v$ . If  $I$  is a closed interval in  $[0, 1]$  we claim that  $L(C_\lambda) \cap I$  coincides with the set  $r(D(v_I))$ , for some unique vertex  $v_I \in \mathcal{T}$ . Furthermore, if  $I_1, I_2, \dots, I_k$  is a set of disjoint standard intervals covering  $C_\lambda$  then  $v_{I_1}, v_{I_2}, \dots, v_{I_k}$  are the leaves of a finite binary subtree

$T(I_1, I_2, \dots, I_k)$  of  $\mathcal{T}$  containing the root. In particular, if  $J_1, J_2, \dots, J_k$  is another covering of  $C_\lambda$  by standard intervals then we have two finite trees  $T(I_1, I_2, \dots, I_k)$  and  $T(J_1, J_2, \dots, J_k)$ . Further, we also have affine bijections  $\varphi_j : I_j \rightarrow J_j$  which are of the form  $\varphi_j(x) = b_j + \lambda^{k_j}(x - a_j)$ , where  $a_j, b_j \in L(C_\lambda)$ . It is clear that  $\varphi_j(I_j \cap L(C_\lambda)) = J_j \cap L(C_\lambda)$ . The explicit form of  $\varphi_j|_{I_j \cap L(C_\lambda)}$  actually can be interpreted in terms of  $r(v_{I_j})$ , as follows. Let  $\mathcal{D}(v)$  be the planar rooted subtree of  $\mathcal{T}$  of vertices  $D(v)$  and root  $v$ . There is a natural identification  $\iota_{v,w}$  of the planar binary rooted trees  $D(v)$  and  $D(w)$ , for any  $v, w \in \mathcal{T}$ . When we further identify  $L(C_\lambda) \cap I_j$  with the set  $r(D(v_{I_j}))$  the induced action of  $\varphi_j$  on  $w \in D(v_{I_j})$  coincides with  $\iota_{v_{I_j}, v_{J_j}}$ .

Consider now the operation of replacing an interval  $I_j$  by two disjoint intervals  $I'_j$  and  $I''_j$  whose union is disjoint from the other intervals  $I_k$ . Correspondingly we replace  $J_j$  by the couple  $\{J'_j, J''_j\} = \{\varphi_j(I'_j), \varphi_j(I''_j)\}$  and  $\varphi_j$  by its restrictions to these smaller intervals. This operation does not change the element in  $\mathfrak{diff}^1(C_\lambda)$ . The immediate consequence of the description of  $\varphi_j$  is that the pairs of trees  $T(I_1, \dots, I'_j, I''_j, \dots, I_k)$  and  $T(J_1, \dots, J'_j, J''_j, \dots, J_k)$  are both obtained from  $T(I_1, I_2, \dots, I_k)$  and  $T(J_1, J_2, \dots, J_k)$  by adding one caret at the  $j$ -th leaf. This proves that this pair of trees is a well-defined element of the standard Thompson group  $F$ . It is rather clear that the map defined this way  $\mathfrak{diff}^{1,+}(C_\lambda) \rightarrow F$  is an isomorphism.

In a similar way we define an isomorphism  $\mathfrak{diff}_{S^1}^{1,+}(C_\lambda) \rightarrow T$ , when we work with the infinite unrooted binary tree  $\mathcal{T}$  embedded in the plane so that its ends abut to  $S^1$ .

In the case of  $\mathfrak{diff}_{S^2}^1(C_\lambda)$  we use the proof of Theorem 3 and the infinite unrooted binary tree  $\mathcal{T}$  without any planar structure. The only difference is that the restrictions  $\varphi|_{I_j}$  are not having anymore a coherent orientation. Some of them might be orientation preserving while the others not. The orientation data is encoded by an element of the infinite sum  $\oplus_1^\infty \mathbb{Z}/2\mathbb{Z} = \bigcup_{n=1}^\infty (\mathbb{Z}/2\mathbb{Z})^n$ . This explains the isomorphism between  $\mathfrak{diff}_{S^2}^1(C_\lambda)$  and the extension  $V^\pm$  of the Thompson group  $V$ . This ends the proof of Theorem 6 in the case of  $C = C_\lambda$ .

### 4.3 Proof of the general case of Theorem 6

The only missing ingredient is the result generalizing Lemma 12 to the more general self-similar sets considered here, as follows:

**Lemma 13.** *Let  $\varphi \in \text{Diff}^{1,+}(\mathbb{R}, C)$ . Then there is a covering of  $C$  by a finite collection of disjoint intervals  $I_k$ , such that  $\varphi|_{C \cap I_k}$  is the restriction of an affine function to  $I_k \cap C$ .*

The proof of this lemma for incommensurable parameters will occupy section 4.3.1. In the case when gaps and homothety factors are respectively equal the proof given above extends word by word.

Now, any  $\varphi$  in the group  $\mathfrak{diff}^{1,+}(C_\Phi)$  corresponds to a pair of coverings of  $C$  by intervals  $(I_1, I_2, \dots, I_k)$  and  $(J_1, J_2, \dots, J_k)$  so that  $\varphi$  sends affinely  $I_j$  into  $J_j$ , for all  $j$ . These intervals could be chosen to be of the form  $[a, b]$ , where  $a$  is a left point of  $C$  and  $b$  is a right point of  $C$ . We call them *clopen* intervals. Particular examples of clopen intervals are the images of  $[0, 1]$  by the semigroup generated by  $\Phi$ , which will be called *standard (clopen) intervals*. Each standard clopen interval corresponds to a finite geodesic path descending from the root in the (regular rooted) tree of valence  $n + 2$  associated to  $\Phi$ . It remains to prove then:

**Lemma 14.** *We assume that  $\Phi$  verifies the genericity condition (C) from Definition 6. Then any  $\varphi \in \mathfrak{diff}^{1,+}(C_\Phi)$  corresponds to a pair of coverings of  $C$  by standard intervals  $(I_1, I_2, \dots, I_k)$  and  $(J_1, J_2, \dots, J_k)$  so that  $\varphi$  sends affinely  $I_j$  into  $J_j$ , for all  $j$ .*

*Proof.* Every clopen interval is the disjoint union of finitely many standard intervals. We can therefore suppose that  $I_j$  are standard intervals.

We claim that the image  $J$  of a standard interval  $I$  by an affine map  $\varphi$  preserving  $C$  must be a standard interval.

We will need in the sequel more terminology. Standard intervals are associated to vertices of the  $(n + 2)$ -valent tree, and one says that they belong to the  $k$ -th *generation of standard intervals* if the associated vertex is at distance  $k$  from the root. The complementary intervals to the union of all  $k$ -th generation of standard intervals will be the  $k$ -th *generation of gaps*. Moreover, given a standard interval  $I$  of the  $k$ -th generation, the gaps of the  $k + 1$ -th generation lying in  $I$  will also be called the first generation of gaps in  $I$ .



Let us write  $J = J^1 \cup J^2 \cup \dots \cup J^m$ , as the union of finitely many standard intervals. Suppose that  $J^u$  is the largest among the intervals  $J^i$ . By further composing  $\varphi$  with the affine map in  $\mathfrak{diff}^{1,+}(C)$  sending  $J^u$  onto  $I$  we can assume that  $I = J^u$ . In particular, the homothety factor  $\mu$  of the affine map  $\varphi : J^u \rightarrow J$  is at least 1.

Assume that  $C$  is central, namely the IFS is homogeneous (i.e. all  $\lambda_i = \lambda$ ) and the initial (i.e. first generation) gaps have the same length. Then the set of largest gaps in  $J^u$  consists of  $n$  equidistant gaps of the same size  $g$ . Their image by an affine map should be the set of largest gaps in  $J$ , so that there are  $n$  equidistant equal gaps in  $J$ . The only possibility for the size of these gaps is  $\lambda^n g$ , for some  $n \in \mathbb{Z}_+$ . Every such image gap determines uniquely a standard interval  $J_n$  of size  $\lambda^n$  to which it belongs. If two of these gaps determined distinct standard intervals, then they would be separated by another gap of size  $\lambda^{n-1}g$ , contradicting their maximality. Then all but possibly the leftmost and rightmost intervals of the complement of these  $n$  gaps in  $J$  are standard.

Now, the interval between two consecutive gaps in  $J^u$  is a standard interval of length  $\lambda$ , whose image by an affine map should have length  $\lambda^{1+n}$ . This shows that the leftmost and the rightmost intervals also should be standard intervals, having the same size as the other  $n-2$  standard intervals between consecutive image gaps. Altogether this shows that  $J = J_n$  is a standard interval.

The set of gaps of the same generation is totally ordered from the leftmost gap towards the right. The sequence of lengths of  $(k+1)$ -th generation gaps within a standard interval of the  $k$ -th generation is of the form  $(\Lambda_{\mathbf{k}}g_1, \Lambda_{\mathbf{k}}g_2, \dots, \Lambda_{\mathbf{k}}g_n)$ , for some  $\mathbf{k}$ . Consider now a gap of the first generation, say  $\Lambda_{\mathbf{k}}g_\alpha$  of  $J^u$ . Its image by an affine map should be a gap of  $J$ . It follows that there exists some  $\sigma(\alpha) \in \{1, 2, \dots, n\}$  and  $\mathbf{k}_\alpha \in \mathbb{Z}_+^{n+1}$ , so that:

$$\mu \Lambda_{\mathbf{k}}g_\alpha = \Lambda_{\mathbf{k}_\alpha}g_{\sigma(\alpha)},$$

where  $\mu$  is the homothety factor of the map  $\varphi$ . Conversely, any gap of  $J^u \subset J$  is the image by  $\varphi$  of some gap of  $J^u$ , and hence there exists some  $\tau(\alpha) \in \{1, 2, \dots, n\}$  and  $\mathbf{l}_\alpha \in \mathbb{Z}_+^{n+1}$ , so that:

$$\frac{1}{\mu} \Lambda_{\mathbf{k}}g_\alpha = \Lambda_{\mathbf{l}_\alpha}g_{\tau(\alpha)}.$$

Getting rid of  $\mu$  in the two equalities above we obtain the following identities, for all  $\alpha, \beta$ :

$$\Lambda_{\mathbf{k}_\alpha + \mathbf{l}_\beta - 2\mathbf{k}} g_{\sigma(\alpha)} g_{\tau(\beta)} = g_\alpha g_\beta.$$

By taking  $\beta = \sigma(\alpha)$  we derive:

$$\Lambda_{\mathbf{k}_\alpha + \mathbf{l}_{\sigma(\alpha)} - 2\mathbf{k}} g_{\tau(\sigma(\alpha))} = g_\alpha.$$

If  $g_\alpha$  and  $\lambda_j$  satisfy the genericity condition (C) the last equality implies  $\tau(\sigma(\alpha)) = \alpha$  and  $\mathbf{k}_\alpha + \mathbf{l}_{\sigma(\alpha)} = 2\mathbf{k}$ , for every  $\alpha$ . A symmetric argument yields  $\sigma(\tau(\alpha)) = \alpha$ , so that  $\sigma$  and  $\tau$  are bijections inverse to each other. Furthermore we derive:

$$\mu^n = \prod_{\alpha=1}^n \Lambda_{\mathbf{k}_\alpha - \mathbf{k}} \frac{g_{\sigma(\alpha)}}{g_\alpha} = \Lambda_{\sum_{\alpha=1}^n (\mathbf{k}_\alpha - \mathbf{k})},$$

so that

$$\mu = \Lambda_{-\mathbf{k} + \frac{1}{n} \sum_{\alpha=1}^n \mathbf{k}_\alpha}.$$

Therefore, for each  $\alpha$  we have:

$$\frac{g_{\sigma(\alpha)}}{g_\alpha} = \Lambda_{-\mathbf{k} + \frac{1}{n} \sum_{\alpha=1}^n \mathbf{k}_\alpha}.$$

Then our assumptions of genericity imply that  $\sigma$  must be identity. It turns that  $\mathbf{k}_\alpha = \mathbf{k}$  and hence  $\mu = 1$ . Therefore  $J = J^u$  and thus  $J$  is standard.  $\square$

Now, it is immediate that  $\langle \lambda_0 \rangle \subset \chi(\text{Diff}_0^{1,+})$ , and by Lemma 7 there exists some  $N \in \mathbb{Z}_+$  so that  $\chi(\text{Diff}_0^{1,+}) = \langle \lambda_1^{1/N} \rangle$ . Since  $L(C)$  is affinely locally homogeneous this holds for any left point  $a$  of  $C$ .

Then, the general form of an affine germ locally preserving  $C$  around a left point  $c_k \in C \cap I_k$  is:

$$\varphi(x) = \varphi(c_k) + \Lambda_{\mathbf{j}_k, N}(x - c_k), \quad \text{for } x \in I_k \cap C, \quad (38)$$

where, for each multi-index  $\mathbf{k} = (k_0, k_1, \dots, k_n)$  we put:

$$\Lambda_{\mathbf{k}, N} = \lambda_0^{k_0/N} \prod_{i=1}^n \lambda_i^{k_i}. \quad (39)$$

Furthermore, we can modify any germ in  $\text{Diff}_0^{1,+}$  by using homotheties of ratios  $\lambda_0^k$ ,  $k \in \mathbb{Z}$  in order to obtain a diffeomorphism  $\varphi : [0, 1] \rightarrow [0, r]$  sending  $C$  into  $C$ . By Lemma 13 we can assume that  $\varphi$  is an affine map, and by Lemma 14  $[0, r]$  must be a standard interval. It follows that the homothety factor of  $\varphi$  is a power of  $\lambda_0$ . This implies that  $N = 1$ .

Pairs of coverings by standard clopen intervals of  $C$  correspond to pairs of finite rooted subtrees. Subdividing the covering by standard subintervals is then equivalent to stabilizing the trees. This provides isomorphisms with the Thompson groups  $F_{n+1}$ ,  $T_{n+1}$  and  $V_{n+1}^\pm$ , respectively, ending the proof of Theorem 6.

#### 4.3.1 Proof of Lemma 13 for incommensurable parameters

We will use a much weaker restriction than the total incommensurability, see below.

Recall the notation from section 4.1 concerning the rooted  $(n+2)$ -valent labeled tree with edges directed downwards, and all edges issued from a vertex being enumerated from left to the right.

With this notation left points of  $C$  correspond to sequences which eventually end in 1, namely of the form

$$L(i_1 \dots i_p) = i_1 i_2 \dots i_p 111111 \dots,$$

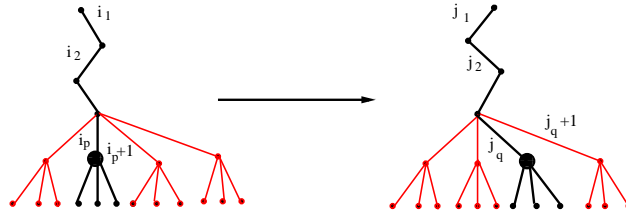
while right points correspond to sequences which eventually end in  $n$ :

$$R(i_1 \dots i_p) = i_1 \dots i_p nnnnnn \dots$$

The standard germs are in this case affine functions, which can be therefore extended to the whole line. Consider two finite multi-indices  $I = i_1 \dots i_p$  and  $J = j_1 \dots j_q$  and set  $a = L(i_1 \dots i_p)$ ,  $b = R(i_1 \dots i_p)$ ,  $\alpha = L(j_1 \dots j_q)$ ,  $\beta = R(j_1 \dots j_q)$ . The *standard germ* associated to these indices is the affine map  $\psi_{I,J} : [a, b] \rightarrow [\alpha, \beta]$  given by the formula:

$$\psi_{I,J}(x) = a + \left( \frac{\prod_{m=1}^q \lambda_{j_m}}{\prod_{k=1}^p \lambda_{i_k}^{-1}} \right) (x - a).$$

Each multi-index  $I$  determine a vertex  $v_I$  of the tree, which is the endpoint of the geodesic issued from the root which travels along the edges labeled  $i_1, i_2, \dots, i_p$ . Then, at the level of trees a standard germ corresponds to a combinatorial map sending the subtree hanging at the vertex  $v_I$  onto the subtree issued from the vertex  $v_J$ , as in the figure below:



An *extension* of the standard germ  $\psi : [a, b] \rightarrow [\alpha, \beta]$  is a standard germ defined on  $[c, d] \supset [a, b]$  whose restriction to  $[a, b]$  coincides with  $\psi$ . In this case  $[c, d]$  must correspond to a vertex  $v_{I'}$  of the tree whose multi-index  $I'$  is a prefix of  $I$ , namely  $I' = i_1 i_2 \dots i_r$  with  $r \leq p$ . This shows that a non-trivial extension of  $\psi$  exists only if  $i_p = j_q$ .

A *multi-germ* is a finite collection of standard germs  $\psi_j : [a_j, b_j] \rightarrow [\alpha_j, \beta_j]$  such that:

$$a_1 < b_1 < a_2 < b_2 < c \dots < a_k < b_k, \quad \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_k < \beta_k$$

and  $[b_j, a_{j+1}]$  and  $[\beta_j, \alpha_{j+1}]$  are gaps of  $C$ , for all  $j$ .

Eventually an *extension of a multi-germ*  $\{\psi_j\}_{j=1,k}$  is a multi-germ  $\{\theta_j\}_{j=1,m}$  such that every standard germ  $\psi_j$  is extended by some  $\theta_i$ . Notice that several elements of the multi-germ  $\{\psi_j\}_{j=1,k}$  might have the same extension  $\theta_i$ .

**Lemma 15.** Let  $\{\psi_j\}_{j=1,m}$  be a chain with the property that there exist constants  $\mu, \nu > 0$  satisfying

$$\frac{\mu}{\nu} > \frac{1}{\max(\lambda_1, \lambda_2, \dots, \lambda_n)},$$

such that:

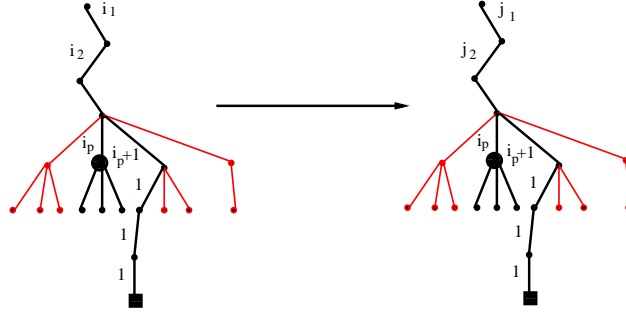
$$\mu \leq \psi'_j(x) \leq \nu, \quad \text{for every } x. \quad (40)$$

If the standard germ  $\psi_j$  admits an extension  $\chi$ , then there exists an extension of the multi-germ  $\{\psi_j\}_{j=1,m}$  containing the standard germ  $\chi$ .

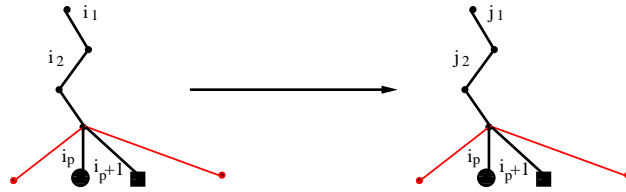
Moreover, if a diffeomorphism  $\varphi \in \text{Diff}^1(\mathbb{R}, C)$  whose derivative  $\varphi'$  verifies the condition for derivative (40) coincides with the multi-germ  $\{\psi_j\}_{j=1,m}$  on  $[a_1, b_m]$ , then it coincides with  $\chi$  on its domain of definiteness.

*Proof.* The standard germ  $\psi_j$  is of the form  $\psi_j = \psi_{I,J}$ , with  $i_p = j_q = k$ . We want to construct an increasing function extending the standard germ  $\psi_{I,J}$  which satisfies the condition (40) for the derivative. Such a function will be called a *continuation* of  $\psi_j$ . Moving one step upward on the tree (i.e. the ancestor vertices) we arrive at the vertices  $v_{I'}$  and  $v_{J'}$ , where  $I = I'k$ ,  $J = J'k$ .

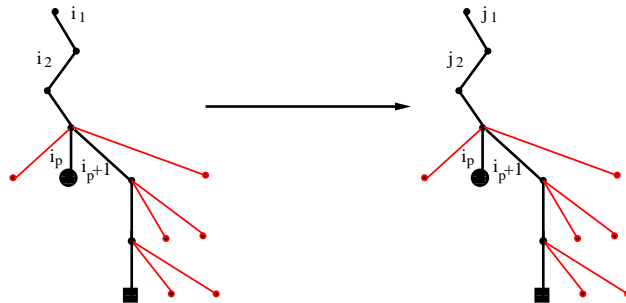
Let, for the sake of definiteness take first  $k < n$  and seek for a continuation on the right side of the interval on which  $\psi_{I,J}$  is defined. Therefore the continuation must have form drawn below, where points marked by squares correspond to each other:



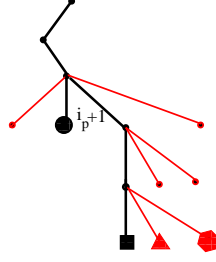
Since the ratio of the derivatives is uniformly bounded, the vertices corresponding to squares should be on the same level, namely at equal distance from the vertices  $v_{I'}$  and  $v_{J'}$ , respectively. Consider the highest possible level of such squares for which the extended map is compatible with the standard germ  $\psi_{j+1}$ . We claim that this continuation has the following form, namely that squares sit on the vertices  $v_{I'k+1}$  and  $v_{J'k+1}$ :



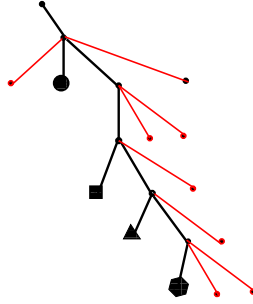
Assume the contrary holds, namely that the squares sit on lower levels, as in the figure below:



Consider further continuation to the right of this increasing function. We label points on the next branch issued from the ancestor of squares vertices by triangles and further by hexagons etc. Consider further the highest levels for which continuation is compatible. Then the picture



is impossible, since then the ancestor of the square vertex also should have been labeled by a square. Therefore we must continue along an infinite path down to a boundary point of the tree, as in the figure:



The boundary point corresponds to an infinite multi-index  $I$ . Then  $\xi = \xi(I) \in [0, 1]$  cannot be a right point of the Cantor set, since this would give a continuation to a whole subtree issued from  $v_{I'}$ , contradicting the form of our path.

Now our continuation coincides with the multi-germ  $\{\psi_j\}_{j=1,m}$  for values  $x \in [a_j, \xi]$ . Since  $\xi$  is not a right point, they coincide in a right semi-neighborhood of  $\xi$  and this contradicts the choice of our infinite path.

We summarize the discussion above as follows. Let  $k_r < n$ . Then the only possible right continuation (which satisfies the condition (40)) of  $\psi_{Ik_1 \dots k_r, Jk_1 \dots k_r}$  is by the germ  $\psi_{Ik_1 \dots k_{r-1}k_r+1, Jk_1 \dots k_{r-1}k_r+1}$ . A similar argument shows that whenever  $k_r > 1$  the only possible left continuation (which satisfies the condition (40)) of  $\psi_{Ik_1 \dots k_r, Jk_1 \dots k_r}$  is by the germ  $\psi_{Ik_1 \dots k_{r-1}k_r-1, Jk_1 \dots k_{r-1}k_r-1}$ .

Repeating the same argument, we get the desired statement.  $\square$

**Lemma 16.** *There exists  $\varepsilon > 0$  with the following property. Consider a standard germ  $\psi_{I,J}$  with  $i_p \neq j_q$  and  $j_q \neq n \neq i_p$ .*

*Then any continuation of  $\psi_{I,J}$  to a standard germ  $\theta$  sending  $L(i_1 i_2 \dots i_{p-1} i_p + 1)$  to  $L(j_1 j_2 \dots j_{q-1} j_q + 1)$  which is defined in a right semi-neighborhood of  $L(i_1 i_2 \dots i_{p-1} i_p + 1)$  is either an extension of the standard germ  $\psi_{I,J}$ , or else it verifies:*

$$\left| \frac{\psi'_{I,J}}{\theta'} - 1 \right| > \varepsilon.$$

Notice that  $\theta$  is locally affine and hence we don't need to specify the point (of the corresponding domain of definiteness) in which we consider the derivative.

*Proof.* The ratio of the derivatives of the standard germs  $\psi_{I,J}$  and  $\theta = \psi_{i_1 i_2 \dots i_{p-1} i_p + 1, j_1 j_2 \dots j_{q-1} j_q + 1}$  is given by:

$$\frac{\psi'}{\theta'} = \frac{\lambda_{i_p}^{-1} \lambda_{i_q}}{\lambda_{i_p+1}^{-1} \lambda_{i_q+1}} \lambda_1^m,$$

where  $m \in \mathbb{Z}$ . This is a discrete subset of  $\mathbb{R}^*$  and hence the claim.  $\square$

We can apply the same arguments when  $i_p \neq 1 \neq j_q$ . Specifically, we have:

**Lemma 17.** *Let  $n \geq 3$ . Then there exists  $\varepsilon > 0$  such that any multi-germ  $\{\psi_j\}_{j=1,m}$  with the property:*

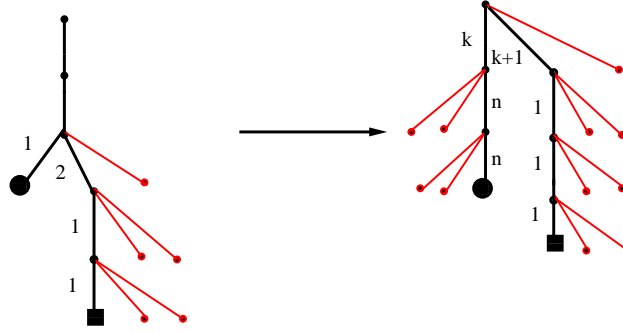
$$\left| \frac{\psi'_i}{\psi'_j} - 1 \right| < \varepsilon$$

*admits an extension containing with at most two elements.*

*Proof.* It remains to examine the standard germs  $\psi_{I,J}$  in following two cases:

$$(I, J) \in \{(i_1 \dots i_{p-1} 1, J = j_1 \dots j_{q-1} n), (i_1 \dots i_{p-1} n, j_1 \dots j_{q-1} 1)\}.$$

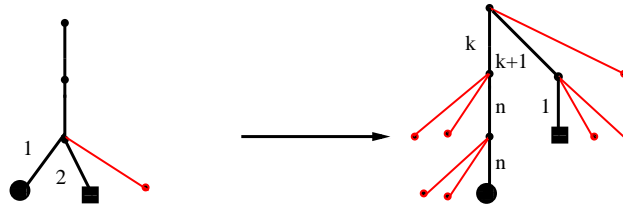
The corresponding picture depends on the number  $s$  of occurrences of  $n$  in the tail of  $j_1 \dots j_{q-1} n$  and the positions of the the square vertices (having  $r$  and  $m$  respectively ancestors labeled 1) as below:



The ratio of derivatives is

$$\frac{\lambda_n^s \lambda_k \lambda_{k+1}^{-1} \lambda_1^{-r}}{\lambda_1 \lambda_2^{-1} \lambda_1^{-m}} = \frac{\lambda_k \lambda_2}{\lambda_{k+1}} \cdot \frac{\lambda_n^s}{\lambda_1^{r-m+1}}$$

Letting  $s$  and  $\mu = r - m + 1$  be large enough we can insure that  $\lambda_n^s / \lambda_1^\mu$  is arbitrarily close to  $\lambda_{k+1} / \lambda_k \lambda_2$ . In this case  $\mu > 0$ , so that we can automatically extend the new standard germ obtained this way and get the figure below, where the position of the squared vertex is the highest possible:



Now, as  $n \geq 3$  we cannot find a non-trivial extension of the two standard germs corresponding to the labeled vertices. This means that there is an extension with at most two elements, thereby proving our statement.  $\square$

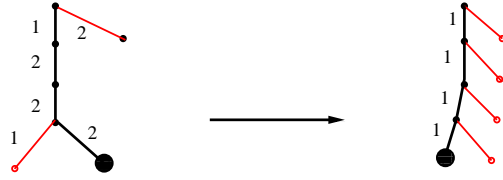
**Lemma 18.** *Let  $n = 2$ . Then there exists  $\varepsilon > 0$  such that any chain  $\{\psi_j\}_{j=1,m}$  verifying the condition:*

$$\left| \frac{\psi'_i}{\psi'_j} - 1 \right| < 1 + \varepsilon$$

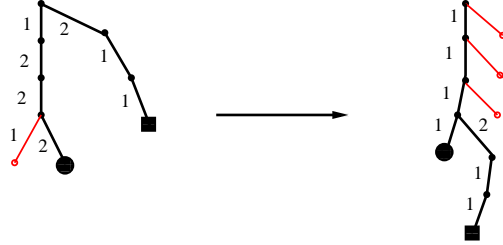
*admits an extension containing at most 4 elements.*

*Proof.* The only possible situation is that pictured below:





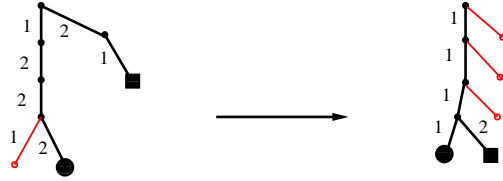
Consider a right continuation as follows:



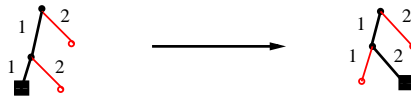
In the left hand side picture we have  $r + 1$  ancestors of the fat dotted vertex which are labeled 2 and  $s$  ancestors of the square vertex labeled 1, while in the right picture there are  $v$  ancestors of the square vertex labeled 1. Then the ratio of derivatives of the two standard germs is:

$$\frac{\lambda_2^{-r-1} \lambda_2 \lambda_1^s}{\lambda_2 \lambda_1^v} = \frac{\lambda_1^{s-v}}{\lambda_2^{r+1}}$$

We can approximate arbitrarily close 1 by  $\lambda_1^{s-v}/\lambda_2^{r+1}$ , but then  $s-v$  must be large, and in particular positive. This implies that we can automatically extend this to a standard germ as follows:



or, after removing nonessential information:



And we now see that a right continuation is impossible. Thus we get our claim.  $\square$

#### 4.3.2 Cantor sets with commensurable parameters

The genericity condition (C) could be extended to include also the case when all homothety factors  $\lambda_i$  are commensurable. We skip the details and present instead an example of an asymmetric Cantor set  $AC$  which is the attractor of the IFS:

$$\phi_0(x) = \frac{1}{4}x, \quad \phi_1(x) = \frac{1}{2}x + \frac{1}{2}.$$

For each finite multi-index  $I = i_1 i_2 \dots i_k$ , with  $i_j \in \{0, 1\}$  we set  $l_\emptyset = 0$  and define by induction:

$$l_{0I} = \frac{1}{4}l_I, \quad l_{1I} = \frac{1}{2}l_I + \frac{1}{2}.$$

Then  $L(AC) = \{l_I; I \text{ finite and admissible}\}$ , where  $I = i_1 i_2 \dots i_k$  is admissible if it is either empty or else  $i_k = 1$ . If we set  $l_I = \lim_{k \rightarrow \infty} l_{i_1 i_2 \dots i_k}$  for infinite  $I$ , then  $AC$  is the union of  $L(AC)$  and the set of  $l_I$ , with

infinite  $I$ . Further,  $L(AC)$  is identified with the set of those  $l_I$  for which  $I$  is infinite and eventually 0. It follows as in the case of  $C_\lambda$  that  $\chi(\text{Diff}_0^1(AC)) = \langle 4 \rangle$ . Further, for  $a, b \in L(AC)$ , we obtain:

$$D(a, b) = \left\{ \psi_{a,b,k} = b + \frac{1}{2^{n(a,b)}} 4^{-k} (x - a), k \in \mathbb{Z} \right\}, \quad (41)$$

where  $n(a, b) \in \{0, 1\}$  is the parity of the length of the geodesic joining  $a$  to  $b$  in the reduced binary tree associated to the IFS.

The previous arguments show that any element  $\varphi$  of  $\mathfrak{diff}^{1,+}(AC)$  determines a *finite* covering of  $AC$  by intervals  $I_j$  on which  $\varphi|_{I_j}$  is of the form  $\psi_{a_j,b_j,k_j}$ , for some  $a_j \in L(AC)$ . Moreover  $\mathfrak{diff}^{1,+}(AC)$  is isomorphic to the Thompson group  $F$ .

#### 4.4 Proof of Theorem 7

Let  $\text{Diff}_a^1(\mathbb{R}^n, C)$  denote the stabilizer of  $a \in C$  in the group  $\text{Diff}^1(\mathbb{R}^n, C)$ . We verify immediately that the map  $\chi : \text{Diff}_a^1(\mathbb{R}^n, C) \rightarrow GL(n, \mathbb{R})$ , given by  $\chi(\varphi) = D_a\varphi$  is a homomorphism. In the case when  $C$  is a product we can describe explicitly  $\chi(\text{Diff}_a^1(\mathbb{R}^n, C))$ . For the sake of simplicity we restrict ourselves to the case  $n = 2$ . Consider  $C = C_{\lambda_1} \times C_{\lambda_2}$ . We say that  $a = (a_1, a_2) \in C$  is an *end* point of  $C$  if both  $a_i$  are endpoints of  $C_{\lambda_i}$ .

**Lemma 19.** *Suppose that  $\lambda_i > 2$  and  $a$  is an end point of  $C$ .*

1. *If  $\lambda_1 \neq \lambda_2$  then*

$$\chi(\text{Diff}_a^1(\mathbb{R}^2, C)) = \langle \lambda_1 \rangle \oplus \langle \lambda_2 \rangle. \quad (42)$$

2. *If  $\lambda_1 = \lambda_2 = \lambda$  then*

$$\chi(\text{Diff}_a^{1,+}(\mathbb{R}^2, C)) = \langle \lambda \rangle \oplus \langle \lambda \rangle. \quad (43)$$

*Proof.* From the first part of the proof of Theorem 4 we infer that whenever  $C$  is a product and  $a \in C$  is fixed by  $\varphi$  its differential  $D_a\varphi$  must send both horizontal and vertical vectors into horizontal or vertical vectors.

Moreover, when  $\lambda_i$  are distinct the horizontality/verticality of the segment should be preserved. Otherwise  $\varphi$  would induce a germ of  $\mathcal{C}^1$ -diffeomorphism  $\phi : (\mathbb{R}, C_{\lambda_1}) \rightarrow (\mathbb{R}, C_{\lambda_2})$ . By Remark 1 we need  $\lambda_1 = \lambda_2$ .

Therefore  $\varphi$  restricts to germs of diffeomorphisms  $\phi_i \in \text{Diff}_{a_i}^1(\mathbb{R}, C_{\lambda_i})$ . By Lemma 11 and Remark 4  $\chi(\phi_i) = \langle \lambda_i \rangle$ . This proves the first item.

On the other hand when  $\lambda_1 = \lambda_2$  we can locally identify  $(C_{\lambda_1}, a_1)$  and  $(C_{\lambda_2}, a_2)$  by an affine germ. The linear map  $R_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  which exchanges the two orthogonal axes meeting at  $a \in C$  belongs to  $\text{Diff}_a^1(\mathbb{R}^2, C)$ . We can compose  $\varphi$  with  $R_a$ , if needed, in order to have  $D_a\varphi$  is diagonal. Thus  $\chi(\text{Diff}_a^1(\mathbb{R}^2, C)) = \langle \langle \lambda \rangle \oplus \langle \lambda \rangle, R_a \rangle$ . Taking into account that  $\det(R_a) = -1$  is not orientation preserving we obtain the claim.

Observe that  $\mathfrak{diff}_{a,\mathbb{R}^2}^1(C)$  is either isomorphic to  $\mathbb{Z}^2$ , when  $\lambda_i$  are distinct, or an extension of  $\mathbb{Z}^2$  by  $\mathbb{Z}/2\mathbb{Z}$ , otherwise.  $\square$

Consider now that  $\lambda_1 = \lambda_2$ . Let now  $a$  and  $b$  be two end points of  $C$ . Denote by  $D(a, b)$  the set of germs at  $a$  of classes in  $\mathfrak{diff}_{a,\mathbb{R}^2}^1(C)$  having representatives  $\varphi \in \text{Diff}^1(\mathbb{R}^2, C)$  such that  $\varphi(a) = b$ . This set is acted upon transitively by  $\mathfrak{diff}_{a,\mathbb{R}^2}^{1,+}(C)$ , so that  $D(a, b)$  consists of maps of the form:

$$\psi_{a,b,\mathbf{k},n}(x) = (b_{j,i} + \lambda^{k_j}(x_i - a_{j,i}))_{i=1,2} \circ S_a^{n_b}, \quad \text{for any } x \in I_j \cap C. \quad (44)$$

where  $S_a^{n_b}$  is an element of the group  $D_2$  of orientation preserving symmetries of the square, namely  $S_a$  is a rotation of order 4 fixing  $a$  and  $n_b \in \{0, 1, 2, 3\}$ .

Now the set of endpoints of  $C$  is kept invariant by any  $\varphi \in \text{Diff}^1(\mathbb{R}^2, C)$ . Therefore, for any endpoint  $a \in C$  there exists some  $k_i, n$  depending on  $a$  such that  $D_a\varphi = (\lambda^{k_1} \oplus \lambda^{k_2}) \circ S_a^n$ . The set of possible values of  $D_a\varphi$  is then a discrete subset of  $GL(2, \mathbb{R})$ . Since endpoints of  $C$  are dense in  $C$  and  $D\varphi$  is continuous we have  $D_a\varphi$  is of the form  $(\lambda^{k_1} \oplus \lambda^{k_2}) \circ S_a^n$ , for any  $a \in C$  and any  $\varphi \in \text{Diff}^{1,+}(\mathbb{R}, C)$ .

For a given  $\varphi$  both the norm  $\|D\varphi\|$  and the determinant  $\det(D\varphi)$  of its differential are continuous on  $[0, 1] \times [0, 1]$  and hence they are bounded. Moreover, the same argument for  $\varphi^{-1}$  shows that these quantities are also bounded from below away from 0, so that  $D\varphi|_C$  can only take finitely many values. The next point is the analogue of Lemma 12 to this situation:

**Lemma 20.** *There is a covering of  $C$  by a finite collection of disjoint standard rectangles  $I_k$  whose images are standard rectangles such that  $\varphi|_{C \cap I_k}$  is the restriction of an affine function and more precisely we have:*

$$\varphi(x) = (\lambda^{j_{k,1}} \oplus \lambda^{j_{k,2}}) \circ S_{b_k}^{m_k}(x - (\alpha_1, \alpha_2)) + \varphi(\alpha_1, \alpha_2), \text{ for } x \in I_k \cap C, \quad (45)$$

where  $\alpha_i$  are left points of  $C_i$ .

*Proof.* We can choose both  $I_k$  and their images to be standard rectangles, as in the case of central Cantor sets  $C_\lambda$ .

Let  $c \in C$ . Then  $D_c\varphi = (\lambda^{j_{k,1}} \oplus \lambda^{j_{k,2}}) \circ S_{b_k}^{m_k}$ , which we denote by  $A$  for simplicity in the proof. We have to prove that there exists a neighborhood  $U$  of  $c$  such that:

$$\varphi(x) = A(x - (\alpha_1, \alpha_2)) + \varphi(\alpha_1, \alpha_2), \quad \text{for } x \in U \cap C. \quad (46)$$

Such neighborhoods will cover  $C$  and we can extract a finite subcovering by clopen sets to get the statement.

This claim is true for endpoints  $a = (\alpha_1, \alpha_2)$  of  $C$ . It is then sufficient to prove that whenever we have a sequence of endpoints  $a_n \rightarrow a_\infty$  contained in a closed rectangle  $U \subset [0, 1]$  and a  $C^1$ -diffeomorphism  $\varphi : U \rightarrow \varphi(U) \subset [0, 1]$  with  $\varphi(C \cap U) \subset C$ , there exists a neighborhood  $U_{a_\infty}$  of  $a_\infty$  and an affine function  $\psi$  such that for large enough  $n$  the following holds:

$$\varphi(x) = \psi(x), \text{ for } x \in C_\lambda \cap U_{a_\infty}.$$

Around each left point  $a_n$  there are affine maps  $\psi_{a_n} : U_{a_n, k_n} \rightarrow [0, 1]$  defining germs in  $D(a_n, c_n)$ , where  $c_n = \varphi(a_n)$ , such that  $c_n$  converge to  $c_\infty = \varphi(a_\infty)$  and

$$\varphi(x) = \psi_{a_n}(x), \text{ for } x \in C_\lambda \cap U_{a_n, k_n}.$$

We can further assume that  $U_{a_n} \cap C_\lambda$  are clopen sets and we can take  $U_{a_n}$  to be standard rectangles  $[\alpha_{n,1}, \beta_{n,1}] \times [\alpha_{n,2}, \beta_{n,2}]$  where  $\beta_{n,i}$  are right points of  $C_i$ .

There is no loss of generality to assume that  $D\psi_{a_n}|_{C \cap U_{a_n}}$  is independent on  $n$ , say it equals  $(\lambda^{m_1} \oplus \lambda^{m_2})S^j$ .

Replacing  $\varphi$  by its inverse  $\varphi^{-1}$  we can also assume that  $m_1 \leq 0$ . Since  $C_1$  is invariant by the homothety of factor  $\lambda$  and center 0, we can further reduce the problem to the case where  $m_1 = 0$ . We can further assume that  $m_2 \leq 0$  by the same trick and finally get rid of the second diagonal component of the differential. Then, by continuity, we have  $D_{a_\infty}\varphi = S^j$ .

Choose  $n$  large enough so that  $\|D_x\varphi(x) - \mathbf{1}\| < \varepsilon$ , for any  $x$  in a square centered at  $a_\infty$  and containing all  $U_{a_n}$ , with  $n$  large enough. where the exact value of  $\varepsilon$  will be chosen later.

Let now consider the maximal standard rectangle of the form  $U' = [\alpha_{n,1}, \beta_1] \times [\alpha_{n,2}, \beta_2]$  to which we can extend  $\psi_{a_n}$  to an affine function which coincides with  $\varphi$  on  $C \cap U'$ .

The endpoint  $(\beta_1, \beta_2)$  belongs to the closure of three maximal rectangular gaps: the rectangle  $Q$  which is opposite to  $U'$  is a product of two gaps, while the other two  $Q_v$  and  $Q_h$  are products of gaps with one (vertical or horizontal) side of  $U'$ . Since  $D_x\varphi$  is close to identity the image of the rectangular gaps are closed to rectangular gaps of approximatively the same sizes. Now, the images by  $\varphi$  of the vertices of  $Q$  are points of  $C$  forming a rectangle, which is itself the product of two gaps. Thus the sizes of this rectangle belong to the set  $\{(\lambda - 2)\lambda^{-n}; n \in \mathbb{Z}_+\} \times \{(\lambda - 2)\lambda^{-n}; n \in \mathbb{Z}_+\}$ . Since the ratios of two different lengths form a discrete set and  $D_x\varphi$  is close to identity, the four points in the image form a rectangle congruent to  $Q$ . A similar argument holds now for the rectangles  $Q_v$  and  $Q_h$ . This implies that  $\psi_{a_n}$  can be extended to an affine function on a strictly larger rectangle, contradicting our assumptions. This proves the claim.  $\square$

This description shows that  $\text{diff}_{\mathbb{R}^2}^{1,+}(C)$  is isomorphic to a  $2V^{sym}$ , namely an extension by  $D_2^\infty$  of Brin's group  $2V$  (see [5]). Here  $D_2$  is the group of the orientation preserving symmetries of the cube, namely the group of orthogonal matrices with integral entries and unit determinant.

**Remark 9.** As in Theorem 3 one can show that  $\mathfrak{diff}_{\mathbb{R}^{n+k}}^{1,+}(C)$ , for  $k \geq 1$ , is an extension of  $nV$  by infinite sum of copies of the full hyperoctahedral group, namely the group of symmetries of the cube.

The proof from above also shows that  $\mathfrak{diff}_{\mathbb{R}^n}^{1,+}(C)$  is isomorphic to  $nV$ , if  $C = C_{\lambda_1} \times C_{\lambda_2} \times \cdots \times C_{\lambda_n}$ , where  $\lambda_i$  are pairwise distinct but commensurable, namely there exists  $\alpha \in \mathbb{R}_+^*$  and  $k_i \in \mathbb{Z}$  such that  $\lambda_i = \alpha^{k_i}$ , for all  $i$ .

## 5 Examples and counterexamples

### 5.1 Nonsparse Cantor sets with uncountable diffeomorphism group

Let  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  be  $C^\infty$ -function satisfying the following conditions:

$$\begin{aligned} h(x) &= 0, & \text{for } 0 \leq x \leq 1, x \geq 2, \\ h(x) &> 0, & \text{for } 1 < x < 2, \\ h'(x) &> -1. \end{aligned}$$

Since the maps  $g_j : [0, 1] \rightarrow [0, 1]$  given by:

$$g_j(x) = x + 2^{-2^j} h(2^j x), \quad j \in \mathbb{Z}_+^* \quad (47)$$

are strictly increasing they are smooth diffeomorphisms of the interval. The support of  $g_j$  is  $[2^{-j}, 2^{-j+1}]$  and hence the diffeomorphisms  $g_j$  pairwise commute. Their derivatives are of the form:

$$g_j'(x) = 1 + 2^{j-2^j} h'(2^j x),$$

and respectively

$$g_j^{(k)}(x) = 2^{kj-2^j} h^{(k)}(2^j x), \quad \text{for } k \geq 2.$$

Consider the group  $R$  consisting of bounded infinite sequences  $\mathbf{m} = m_1, m_2, \dots$  of integers, endowed with the term-wise addition.

There is a map  $\Theta : R \rightarrow \text{Diff}^0([0, 1])$  given by:

$$\Theta(\mathbf{m}) = \lim_{n \rightarrow \infty} g_1^{m_1} \circ g_2^{m_2} \circ \cdots \circ g_n^{m_n}, \quad (48)$$

where  $g_j^m$  is the  $m$ -fold composition of  $g_j$ . The order in the previous definition does not matter, as the maps commute. The limit map  $\Theta(\mathbf{m})$  is immediately seen to be a homeomorphism of  $[0, 1]$  which is a diffeomorphism outside 0.

Let first consider only those  $\mathbf{m}$  where  $m_j \in \{0, 1\}$ . Then we can compute first:

$$\lim_{x \rightarrow 0} \Theta(\mathbf{m})'(x) = 1,$$

and then

$$\lim_{x \rightarrow 0} \Theta(\mathbf{m})^{(k)}(x) = 1, \quad \text{for } k \geq 2.$$

Therefore  $\Theta(\mathbf{m})$  is a  $C^\infty$  diffeomorphism of  $[0, 1]$ .

Moreover any element of  $R$  can be represented as a product of  $\Theta(\mathbf{m})$ , with  $\mathbf{m}$  of having only 0 or 1 entries. This implies that  $\Theta(R) \subset \text{Diff}^\infty([0, 1])$ . Furthermore it is clear that  $\Theta$  is injective, by looking at the factor corresponding to the first place where two sequences disagree. This implies that  $\Theta$  provides a faithful  $C^\infty$  action of  $R$  by  $C^\infty$  diffeomorphisms on  $[0, 1]$ .

The dynamics of each  $g_j$  on its support  $[2^{-j}, 2^{-j+1}]$  is of type north-south with repelling and attracting fixed points on the boundary. Pick up some  $a_j \in (2^{-j}, 2^{-j+1})$ , so that  $b_j = g_j(a_j) > a_j$ . Then the intervals  $g_j^n((a_j, b_j))$  are all pairwise disjoint. If  $C_j^0 \subset [a_j, b_j]$  is some Cantor set, then the closure of its orbit, namely  $C_j = \bigcup_{k=-\infty}^{\infty} g_j^k(C_j^0) \cup \{2^{-j}, 2^{-j+1}\}$  is a  $g_j$ -invariant Cantor subset of  $[2^{-j}, 2^{-j+1}]$ . Moreover, for any  $n \neq 0$  the restriction  $g_j^n|_{C_j}$  cannot be identity, since  $g_j^n$  is strictly increasing.

Then their union  $C = \bigcup_{j=1}^{\infty} C_j$  is a Cantor subset of  $[0, 1]$  and for  $\mathbf{m}$  not identically 0 we also have  $\Theta(\mathbf{m})|_C$  is not identity. This shows that the diffeomorphism group  $\mathfrak{diff}^\infty(C)$  contains  $R$ . In particular,  $\mathfrak{diff}^\infty(C)$  is uncountable.

## 5.2 Nonsparse Cantor set with trivial diffeomorphism group

Let  $X$  be obtained by removing a sequence of intervals, as follows. At the first step we remove from  $[0, 1]$  the central interval of length  $\frac{1}{4}$ . At the  $m$ -th step we have  $2^m$  intervals which we label, starting from the leftmost to the rightmost as  $I_1^{(m)}, I_2^{(m)}, \dots, I_{2^m}^{(m)}$ . We remove then from  $I_j^{(m)}$  the central interval of length  $2^{-2^{m-1}-1+j}$ . The result of this procedure is a Cantor set  $X$  which is not sparse.

We claim that  $\text{diff}^{1,+}(X) = 1$ . Let first consider a point of  $X$  which is not a right point, for instance 0 and  $\varphi \in \text{Diff}_0^1(\mathbb{R}, X)$ . If  $\varphi'(0) \neq 1$  we can assume without loss of generality that  $\varphi'(0) < 1$ . Consider a sequence of gaps  $(x_n, y_n)$  approaching 0. We have either  $\varphi(x_n, y_n) = (x_n, y_n)$  for all large enough  $n$ , or else  $\varphi(x_n, y_n)$  is a different gap than  $(x_n, y_n)$ .

If there exist infinitely many  $n$  such that the gap  $\varphi(x_n, y_n)$  either coincides or is on the right side of  $(x_n, y_n)$  we have  $\varphi(x_n) \geq x_n$  and taking the limit or  $n \rightarrow \infty$  we would obtain  $\varphi'(0) \geq 1$ , contradicting our assumptions. Thus we can suppose that for all  $n$  the gap  $\varphi(x_n, y_n)$  is different from  $(x_n, y_n)$  and lies to its left side. Moreover, for large enough  $n$  we have

$$\left| \frac{\varphi(x_n) - \varphi(y_n)}{x_n - y_n} \right| < 1,$$

since otherwise we would obtain as above  $\varphi'(0) \geq 1$ . Now lengths of gaps belong to the discrete set  $\{2^{-2^n}, n \in \mathbb{Z}_+\}$  and there are not two gaps of the same length. Thus if  $|x_n - y_n| = 2^{-2^{a_n}}$ , for some increasing sequence  $a_n$  of integers then  $|\varphi(x_n) - \varphi(y_n)| \leq 2^{-2^{1+a_n}}$ . Therefore

$$\left| \frac{\varphi(x_n) - \varphi(y_n)}{x_n - y_n} \right| \leq 2^{-2^{1+a_n}+2^{a_n}}.$$

Taking  $n \rightarrow \infty$  we derive that  $\varphi'(0) = 0$ , which contradicts the fact that  $\varphi$  was a diffeomorphism. This proves that  $\text{diff}_0^{1,+}(X) = 1$ .

Let now  $a \in X$ , with  $a \neq 0$  and some germ  $\varphi \in \text{Diff}^1(\mathbb{R}, X)$  with  $\varphi(0) = a$ . As above we can suppose that  $\varphi'(0) \leq 1$ . Take a sequence of gaps  $(x_n, y_n)$  of length  $|x_n - y_n| = 2^{-2^{a_n}}$ , approaching 0 with increasing  $a_n$ . For infinitely many  $n$  the length of the image gap  $|\varphi(x_n) - \varphi(y_n)|$  is smaller than  $2|x_n - y_n|$ , as otherwise  $\varphi'(0) \geq 2$ . It follows that for  $a_n \geq 2$  we should have

$$\left| \frac{\varphi(x_n) - \varphi(y_n)}{x_n - y_n} \right| \leq 2^{-2^{1+a_n}+2^{a_n}}.$$

which leads again to a contradiction. This argument was not specific to  $0 \in X$  and is valid for any point of  $X$ .

This shows that the only possibility left is that  $\varphi$  is identity.

## 5.3 Sparse Cantor set with trivial diffeomorphism group

Start from the interval  $I^{(0)} = [0, 1]$  by removing a central gap  $J_1^{(1)}$  of size  $(1 - \varepsilon)$ . By recurrence at the  $n$ -th step we have  $2^n$  intervals  $I_j^{(n)}$ ,  $j = 1, \dots, 2^n$ , numbered from the left to the right. To go further we remove a central gap  $J_j^{(n+1)}$  from  $I_j^{(n)}$  of length  $|J_j^{(n+1)}| = (1 - \varepsilon^n)|I_j^{(n)}|$ . The set so obtained is obviously a sparse Cantor set  $C_0$ .

Let  $a \in C_0$ . Let  $b_n$  be the right endpoint of the interval  $I_{j_n}^{(n)}$  to which  $a$  belongs. Then set  $(x_n, y_n)$  for the gap  $J_{j_n}^{(n+1)} \subset I_{j_n}^{(n)}$ . There is no loss of generality of assuming that  $a < x_n < y_n < b_n$ . Given  $\varphi \in \text{Diff}_a^1(\mathbb{R}, C_0)$ , with  $\varphi'(a) \neq 1$ , there are infinitely many  $n$  for which the gap  $J_{j_n}^{(n+1)}$  is not fixed by  $\varphi$ . It follows that either  $\varphi(y_n) < x_n$ , or  $\varphi(x_n) > y_n$ , for infinitely many  $n$ . By symmetry we can assume that the second alternative holds. Then

$$\frac{\varphi(x_n) - x_n}{x_n - a} \geq \frac{|y_n - x_n|}{|x_n - a|} \geq \frac{(1 - \varepsilon^n)|b_n - a|}{|x_n - a|} \geq \frac{(1 - \varepsilon^n)|b_n - a|}{\varepsilon^n|b_n - a|} = \frac{1 - \varepsilon^n}{\varepsilon^n}. \quad (49)$$

Letting  $n \rightarrow \infty$  we obtain that  $\varphi'(a) = \infty$ , contradiction. This proves that  $\text{diff}_a^{1,+}(C_0) = 1$ .

From Remark 5 we have  $\text{diff}^{1,+}(C_0) = 1$ .

**Another potential example.** In order to convert the nonsparse example above  $X$  into a sparse Cantor set with the same properties, we have to mix ordinary gaps and very small gaps. Start as above from the interval  $I^{(0)} = [0, 1]$  by removing a central gap  $LG^{(1)}$  of size  $\frac{1}{3}$  and two very small gaps each one centered within an interval component of  $I^{(0)} - LG^{(1)}$ , namely  $SG_1^{(1)}$  and  $SG_2^{(1)}$  of lengths  $2^{-2^\alpha}$  and  $2^{-2^{\alpha+1}}$ , respectively. Here  $\alpha$  is chosen so that

$$\frac{1}{3} - 2^{-2^\alpha} > \frac{1}{6}.$$

We obtain at the next stage four intervals  $I_1^{(1)}, I_2^{(1)}, I_3^{(1)}, I_4^{(1)}$ , labeled from the left to the right.

By recurrence at the  $n$ -th step we have  $4^n$  intervals  $I_j^{(n)}$ ,  $j = 1, \dots, 4^n$ . To go further we remove first a central gap  $LG_j^{(n+1)}$  from  $I_j^{(n)}$  of length  $|LG_j^{(n+1)}| = \frac{1}{3}|I_j^{(n)}|$ . Further we remove two very small gaps each one centered within an interval component of  $I_j^{(n)} - LG_j^{(n)}$ , namely  $SG_{2j+1}^{(n)}$  and  $SG_{2j+2}^{(n)}$  of lengths  $2^{-2^{\alpha+j+4^n}}$  and  $2^{-2^{\alpha+j+1+4^n}}$ . Letting  $n$  go to  $\infty$  we obtain a  $\frac{1}{3}$ -sparse Cantor set  $MC$ . We believe that  $\mathfrak{diff}^{1,+}(MC) = 1$ .

## 5.4 Split Cantor sets

Two Cantor sets  $C_i \subset \mathbb{R}^n$  are *locally smoothly nonequivalent* if for any  $p_i \in C_i$  there is no  $\mathcal{C}^1$ -diffeomorphism germ  $(\mathbb{R}^n, C_1, p_1) \rightarrow (\mathbb{R}^n, C_2, p_2)$ .

A Cantor set in  $C \subset \mathbb{R}^n$  is said to be *smoothly split* if we can write  $C = C_1 \cup C_2$  as a union of two Cantor sets with  $C_1$  and  $C_2$  locally smoothly nonequivalent and contained in disjoint intervals.

We have the following easy:

**Proposition 2.** *Let  $n \geq 1$  and  $C \subset \mathbb{R}^n$  be a Cantor set which is smoothly split as  $C_1 \cup C_2$ . Then  $\mathfrak{diff}^{1,+}(C) = \mathfrak{diff}^{1,+}(C_1) \times \mathfrak{diff}^{1,+}(C_2)$ .*

*Proof.* In this situation  $C_i$  are contained into disjoint intervals  $U_i$ . Then diffeomorphisms preserving  $C$  should also send each  $C_i$  into itself. Furthermore all elements from  $\mathfrak{diff}^{1,+}(C_1) \times \mathfrak{diff}^{1,+}(C_2)$  can be realized as classes of pairs of commuting diffeomorphisms supported in  $U_i$ .  $\square$

According to Remark 1 the central Cantor sets  $C_\lambda$  are pairwise locally smoothly nonequivalent. In particular the union  $C_\lambda \cup 2 + C_\mu$  of two distinct Cantor sets, one of which is translated by 2 is a split Cantor set. It follows that

$$\mathfrak{diff}^{1,+}(C_\lambda \cup 2 + C_\mu) = \mathfrak{diff}^{1,+}(C_\lambda) \times \mathfrak{diff}^{1,+}(C_\mu) \cong F \times F,$$

for distinct  $\lambda$  and  $\mu$ .

It is not yet clear what would be  $\mathfrak{diff}^{1,+}(C_\lambda \cup C_\mu)$ .

## References

- [1] R. Bamón, C.G. Moreira, S. Plaza, J. Vera, Differentiable structures of central Cantor sets, *Ergodic Theory Dynam. Systems* 17 (1997), no. 5, 1027–1042.
- [2] R. Bieri and R. Strebel, Groups of PL homeomorphisms of the real line, preprint, Math. Sem. de Univ. Frankfurt, Frankfurt am Main, 1985.
- [3] W.A. Blankenship, Generalization of a construction of Antoine, *Ann. of Math.* (2) 53(1951), 276–297.
- [4] C. Bleak and D. Lanoue, A family of non-isomorphism results, *Geom. Dedicata* 146 (2010), 21–26.
- [5] M. G. Brin, Higher dimensional Thompson groups, *Geom. Dedicata* 108 (2004), 163–192.
- [6] M. G. Brin, The algebra of strand splitting. I. A braided version of Thompson’s group V, *J. Group Theory* 10 (2007), no. 6, 757–788.
- [7] M. G. Brin, On the baker’s map and the simplicity of the higher dimensional Thompson groups  $nV$ , *Publ. Mat.* 54 (2010), 433–439.



- [8] K. S. Brown and R. Geoghegan, An infinite-dimensional torsion-free  $\mathrm{FP}_\infty$  group, *Invent. Math.* 77 (1984), no. 2, 367–381.
- [9] K.S. Brown, Finiteness properties of groups, *J.Pure Appl. Algebra* 44 (1987), 45–75.
- [10] J . W. Cannon, W. J. Floyd and W. R. Parry, Introductory notes in Richard Thompson’s groups, *Enseign. Math.* 42 (1996), 215–256.
- [11] D. Cooper and T. Pignataro, On the shape of Cantor sets, *J. Differential Geom.* 28 (1988), no. 2, 203–221.
- [12] P. Dehornoy, The group of parenthesized braids, *Adv. Math.* 205 (2006), no. 2, 354–409.
- [13] B. Deroin, V. Kleptsyn and A. Navas, Sur la dynamique unidimensionnelle en régularité intermédiaire, *Acta Math.* 199 (2007), no. 2, 199–262.
- [14] J.J. Dijkstra and J. van Mill, Erdős space and homeomorphism groups of manifolds, *Mem. Amer. Math. Soc.* 208 (2010), no. 979, vi+62 pp. ISBN: 978-0-8218-4635-3
- [15] K.J.Falconer, *Fractal geometry: Mathematical Foundations and Applications*, John Wiley and Sons, 2003.
- [16] K.J.Falconer and D.T. Marsh, On the Lipschitz equivalence of Cantor sets, *Mathematika* 39(1992), 223–233.
- [17] L. Funar and C. Kapoudjian, On a universal mapping class group of genus zero, *Geom. Funct. Analysis* 14 (2004), 965–1012.
- [18] L. Funar and C. Kapoudjian, The braided Ptolemy-Thompson group is finitely presented, *Geom. Topol.* 12 (2008), 475–530.
- [19] É. Ghys and V. Sergiescu, Sur un groupe remarquable de difféomorphismes du cercle, *Comment. Math. Helv.* 62 (1987), 185–239.
- [20] P.Greenberg and V. Sergiescu, An acyclic extension of the braid group, *Comment. Math. Helv.* 66(1991), 109–138.
- [21] J. Hennig and F. Matucci, Presentations for the higher-dimensional Thompson groups  $nV$ , *Pacific J. Math.* 257 (2012), 53–74.
- [22] G. Higman, Finitely presented infinite simple groups, Notes on Pure Math. 8 (1974), Australian Nat. Univ., Canberra.
- [23] J.E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* 30(1981), 713–747.
- [24] C. Kapoudjian et V. Sergiescu, An extension of the Burau representation to a mapping class group associated to Thompson’s group  $T$ , *Geometry and dynamics*, Amer. Math. Soc. , Vol. 389 , 141–164 (2005).
- [25] G. Laget, Groupes de Thompson projectifs en genre 0, PhD Thesis, Univ. Joseph-Fourier - Grenoble I, Jul. 2004 (French) available at: <http://tel.archives-ouvertes.fr/docs/00/04/71/79/PDF/tel-00006388.pdf>
- [26] D.R. McMillan Jr., Taming Cantor sets in  $E^n$ , *Bull Amer. Math. Soc.* 70(1964), 706–708.
- [27] A. Navas, On uniformly quasisymmetric groups of circle diffeomorphisms, *Ann. Acad. Sci. Fenn. Math.* 31 (2006), 437–462.
- [28] Y.Neretin, Combinatorial analogues of the group of diffeomorphisms of the circle, *Izv. Ross. Akad. Nauk Ser. Mat.* 56 (1992), no. 5, 1072–1085; translation in *Russian Acad. Sci. Izv. Math.* 41 (1993), no. 2, 337–349.
- [29] Hui Rao, Huo-Jun Ruan and Yang Wang, Lipschitz equivalence of Cantor sets and algebraic properties of contraction ratios. *Trans. Amer. Math. Soc.* 364 (2012), no. 3, 1109–1126.
- [30] Matatyahu Rubin, Locally Moving Groups and Reconstruction Problems, Ordered Groups and Infinite Permutation Groups. pp. 121–157. Kluwer Academic Publisher, Dordrecht (1996)
- [31] Matatyahu Rubin, On the reconstruction of topological spaces from their groups of homeomorphisms. *Trans. Amer. Math. Soc.* 312 (1989), no. 2, 487–538.
- [32] A.C. Shilepsky, A rigid Cantor set in  $\mathbf{E}^3$ , *Bull. Acad. Polon. Sci.* 22 (1974), 223–224.
- [33] M. Stein, Groups of piecewise linear homeomorphisms, *Trans. Amer. Math. Soc.* 332 (1992), no. 2, 477–514.

- [34] D. Sullivan, Differentiable structures on fractal-like sets, determined by intrinsic scaling functions on dual Cantor sets, *The mathematical heritage of Hermann Weyl* (Durham, NC, 1987), 15–23, Proc. Sympos. Pure Math., 48, Amer. Math. Soc., Providence, RI, 1988.
- [35] W.P. Thurston, A generalization of the Reeb stability theorem. *Topology* 13 (1974), 347–352.
- [36] D.G. Wright, Rigid sets in  $\mathbf{E}^n$ , *Pacific J. Math.* 121 (1986), 245–256.
- [37] Li-Feng Xi and Ying Xiong, Lipschitz Equivalence Class, Ideal Class and the Gauss Class Number Problem, arXiv:1304.0103.